ON GRAPH AND CUNTZ-KRIEGER TYPE C^*-ALGEBRAS

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Abstract. We clarify the relationship between the Cuntz-Krieger type C^*-algebras, introduced by the first named author in [4, 3], and C^*-algebras associated with higher-rank graphs. In particular we derive the extent to which the two classes of C^*-algebras coincide, thereby enabling the independently developed theories for both classes of C^*-algebras to benefit from one another.

1. Introduction

In seminal work Cuntz, and subsequently Cuntz and Krieger, introduced the Cuntz algebras, and Cuntz-Krieger algebras respectively, as C^*-algebras generated by a system of generators and relations [6, 7, 8]. An important characteristic of these C^*-algebras is that they are canonically unique, in the sense that given another set of generators satisfying the same relations the resulting C^*-algebras are canonically *-isomorphic.

In [2] the first named author sought to prove an analogous uniqueness theorem in a much more general setting. Based largely on the algebraic approach adopted by Cuntz in [6] the uniqueness theorem is shown to hold for any C^*-algebra generated by a system of generators and relations that satisfies three conditions (A), (B) and (C). Examples of such C^*-algebras include not only the motivating Cuntz-Krieger algebras but also to almost all of the C^*-algebras associated to infinite matrices constructed by Exel and Laca in [9] (the Exel-Laca algebras). Further work in this direction was presented in [4] where condition (C) was replaced by a simplified and weaker condition (C'), which enabled the enlarged class of so-called *Cuntz-Krieger type algebras* to contain all Exel-Laca algebras.

Other examples of Cuntz-Krieger type algebras were introduced in [3], which can be thought of as higher rank Cuntz-Krieger type algebras as their definition depends not only on a single matrix, as is the case for the original Cuntz-Krieger algebras, but on a (finite or *infinite*) family of matrices. Work had begun on these higher rank Cuntz-Krieger type algebras before the first named author became aware of a similar construction, namely Robertson and Steger's higher rank Cuntz-Krieger algebras [14]. In this paper we shall consider a larger class of C^*-algebras than the class of Robertson-Steger algebras[1], namely the class of Sims' *relative Cuntz-Krieger algebras of finitely aligned higher-rank graphs* [15]. The relative Cuntz-Krieger algebras of finitely aligned higher-rank graphs are a generalisation of Kumjian and Pask's higher-rank graph C^*-algebras [11], which were constructed to provide a graphical model for Robertson-Steger algebras in analogy with the model that graph C^*-algebras provide for Cuntz-Krieger algebras (see [12] for a comprehensive account of the theory of graph C^*-algebras).

The purpose of this paper is to clarify the relationship between these constructions, which will enable the theories that have been independently developed for

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[1]We shall refer to the higher rank Cuntz-Krieger algebras constructed by Robertson and Steger in [14] as Robertson-Steger algebras.

both types of construction to benefit from each other. An immediate consequence is that the uniqueness theorem for relative Cuntz-Krieger algebras of finitely aligned higher-rank graphs [15] is proved for (potentially) more general k-graphs (see Theorem 2.14). In a forthcoming paper [5] we shall exploit the relationships that we will identify in this paper to investigate the implications for such aspects as the purely infiniteness, ideal structure and K-theory.

The remainder of the paper is organised as follows. In §2 we represent the relative Cuntz-Krieger algebra of a finitely aligned higher-rank graph that satisfies an aperiodicity condition as a Cuntz-Krieger type algebra. In §3 we show that some higher rank Cuntz-Krieger type algebras may be represented as higher-rank graph C^*-algebras when the defining family of matrices and all its constituent matrices are finite.

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2. A representation of higher-rank graph C^*-algebras as Cuntz-Krieger type algebras

Let $(\mathbf{\Lambda}, d)$ be a finitely aligned k-graph [13].[2] Let $\Sigma := \bigcup_{i=1}^k \Lambda^{e_i}$ where $\{e_i\}_{i=1}^k$ are the canonical generators of $\mathbb{N}^k$ as a semi-group.

We state the following definition from [15] (referring the reader to [15, 16] for notation).

Definition 2.1 ([15, Definition 3.2])**.** Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. A *relative Cuntz-Krieger* $(\Lambda;\mathcal{E})$*-family* is a collection $\{t_\lambda \mid \lambda \in \Lambda\}$ of partial isometries[3] in a *-algebra satisfying:

(TCK1) $\{t_v \mid v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(TCK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = s(\mu)$;
(TCK3) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$; and
(CK) $\prod_{\lambda\in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$ for all $E \in \mathcal{E}$.

Remark 2.2.
(1) We note that the original definition of a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family required the partial isometries to lie in a C^*-algebra rather than a *-algebra. We allow for this more general scenario since we will be considering *-algebras with no pre-equipped norms.
(2) For each finitely aligned k-graph Λ, and each subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$ there exists a C^*-algebra $C^*(\Lambda;\mathcal{E})$ generated by a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family $\{s_{\mathcal{E}}(\lambda) \mid \lambda \in \Lambda\}$ which is universal in the sense that if $\{t_\lambda \mid \ \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family in a C^*-algebra B, then there exists a unique homomorphism $\pi : C^*(\Lambda;\mathcal{E}) \longrightarrow B$ such that $\pi(s_{\mathcal{E}}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$.

Recall the following key definitions from [15] (again we follow the notation used in [15]).

[2]We regard a small countable category $\mathcal{C}$ as a sextuple $(\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}), r, s, \circ, \mathrm{Mor}_0(\mathcal{C}))$ where $\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C})$ are countable sets, r and s are the codomain, domain maps respectively, $\circ$ is an associative (partial) composition on $\mathrm{Mor}(\mathcal{C})$ (compatible with r, s) and $\mathrm{Mor}_0(\mathcal{C})$ is a distinguished subset of $\mathrm{Mor}(\mathcal{C})$ called the set of unit morphisms of $\mathcal{C}$. In this notation we have $\mathbf{\Lambda} := (\Lambda^0, \Lambda, r, s, \circ)$. From this point on we shall follow the usual convention of letting Λ denote both the category and the set of morphisms.

[3]We call an element x in a *-algebra a *partial isometry* if $x = xx^*x$.

Definition 2.3 ([15, Definition 6.2])**.** Let (Λ, d) be a k-graph, and let $x : \Omega_{k,d(x)} \longrightarrow \Lambda$ and $y : \Omega_{k,d(y)} \longrightarrow \Lambda$ be graph morphisms. We say that a graph morphism $z : \Omega_{k,d(z)} \longrightarrow \Lambda$ is a *minimal common extension* of x and y if it satisfies

(1) $d(z)_j = \max\{d(x)_j, d(y)_j\}$ for $1 \leq j \leq k$; and
(2) $z|_{\Omega_{k,d(x)}} = x$ and $z|_{\Omega_{k,d(y)}} = y$.

We write $\mathrm{MCE}(x, y)$ for the collection of minimal common extensions of x and y.

Definition 2.4. Let (Λ, d) be a k-graph and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. We say that Λ satisfies property

(AP) if for all $v \in \Lambda^0$ there exists $x \in v\Lambda^*$ satisfying

(1) $$\text{for distinct } \lambda, \mu \in \Lambda r(x), \text{ we have } \mathrm{MCE}(\lambda x, \mu x) = \emptyset;$$

(B) if for all $v \in \Lambda^0$ there exists $x \in v\Lambda^{\leq\infty}$ satisfying (1); and
(C0) if for all $v \in \Lambda^0$ there exists $x \in v\partial(\Lambda; \mathcal{E})$ satisfying (1).

We say that $(\Lambda, \mathcal{E})$ satisfies **(C)** if Λ satisfies **(C0)** and for all $v \in \Lambda^0$, $F \in v\,\mathrm{FE}(\Lambda)\backslash\bar{\mathcal{E}}$ there exists $x \in v\partial(\Lambda; \mathcal{E})\backslash F\partial(\Lambda; \mathcal{E})$ satisfying (1).

Remark 2.5.

(1) Property **(B)** was defined in an equivalent way in [13, Definition 2.8] (see [16, Remark 4.6.7]). Property **(C)** was defined in [15, Theorem 6.3] and [16, Theorem 4.5.2].
(2) In general we have
$$\mathbf{(B)} \implies \mathbf{(AP)} \Leftarrow \mathbf{(C0)} \Leftarrow \mathbf{(C)}.$$
We also have $\Lambda^{\leq\infty} \subseteq \partial(\Lambda; \mathrm{FE}(\Lambda)) \subseteq \Lambda^*$. Therefore, when $\mathcal{E} = \mathrm{FE}(\Lambda)$ we have
$$\mathbf{(B)} \implies \mathbf{(C0)} \iff \mathbf{(C)} \implies \mathbf{(AP)}.$$

Fix a finitely aligned k-graph Λ and a subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$. Let $\mathcal{A} := \{\tilde{t}_\lambda \mid \lambda \in \Sigma\} \sqcup \{\tilde{t}_v \mid v \in \Lambda^0\}$, i.e. an alphabet of symbols indexed $\Sigma \cup \Lambda^0$. Let $\mathbb{F}$ be the free *-algebra generated by $\mathcal{A}$. Let $\mathbb{I} := \ker \pi$, a self-adjoint, two-sided ideal in $\mathbb{F}$, where $\pi : \mathbb{F} \longrightarrow C^*(\Lambda; \mathcal{E})$ is the unique *-homomorphism satisfying $\pi(\tilde{t}_\lambda) = s_{\mathcal{E}}(\lambda)$ for all $\lambda \in \Sigma \cup \Lambda^0$. To avoid confusion, let $t_\lambda := \tilde{t}_\lambda + \mathbb{I} \in \mathbb{F}/\mathbb{I}$ for all $\lambda \in \Sigma \cup \Lambda^0$. There is, of course, a *-monomorphism $\Psi : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\Lambda; \mathcal{E})$ sending t_λ to $s_{\mathcal{E}}(\lambda)$ for all $\lambda \in \Sigma \cup \Lambda^0$.

Let $\theta : \mathbb{T}^k \longrightarrow \mathbb{T}^{\mathcal{A}}$ be defined by
$$\theta(z)_{\tilde{t}_\lambda} = \begin{cases} z_i & \text{if } d(\lambda) = e_i, \\ 1 & \text{if } d(\lambda) = 0, \end{cases}$$
for all $\lambda \in \Sigma \cup \Lambda^0$. It is straightforward to show that θ is a topological group isomorphism of $\mathbb{T}^k$ onto $H := \theta(\mathbb{T}^k)$ and thus in particular H is a closed subgroup of $\mathbb{T}^{\mathcal{A}}$.

We aim to show that conditions (A),(B),(C') from [2] hold for the system $(\mathbb{F}, \mathbb{I}, H)$. For convenience we restate each property using slightly different notation. We shall follow [2] for the remaining notation.

Recall the following distinguished subsets of $\mathbb{F}/\mathbb{I}$:
$$\begin{aligned}
W &:= \{a_1 \cdots a_n \mid n \geq 1,\ a_i \in \mathcal{A} \sqcup \mathcal{A}^* \text{ for } 1 \leq i \leq n\} + \mathbb{I}, \\
W_0 &:= \{w \in W\backslash\{0\} \mid \mathrm{bal}(w) = 0\}, \\
\Delta &:= \{ww^* \mid w \in W\}, \\
\mathbb{A} &:= \mathrm{Alg}^*(W_0) \subseteq \mathbb{F}/\mathbb{I}, \\
\mathbb{A}_0 &:= \mathrm{Alg}^*(\Delta) \subseteq \mathbb{F}/\mathbb{I}, \\
\mathbb{P} &:= \{p \in \mathbb{A} \mid p = p^* = p^2 \neq 0\}, \\
\mathbb{P}_0 &:= \{p \in \mathbb{A}_0 \mid p = p^* = p^2 \neq 0\},
\end{aligned}$$

(see later for definition of bal). There is, of course, a natural partial order on $\mathbb{P}$, which we denote by $\leq$, given by

$$p \leq q \iff pq = p \text{ for all } p, q \in \mathbb{P}.$$

Moreover, for $p, q \in \mathbb{P}$ we write $p \preceq q$ in $\mathbb{A}$ when there exists an element $s \in \mathbb{A}$ such that $ss^*s = s$, $s^*s = p$ and $ss^* \leq q$.

Definition 2.6. We say that the system $(\mathbb{F}, \mathbb{I}, H)$ satisfies property

(A) if $\Gamma_z(\mathbb{I}) \subseteq \mathbb{I}$ for all $z \in H$, where $\Gamma_z : \mathbb{F} \longrightarrow \mathbb{F}$ is the *-automorphism satisfying $\Gamma_z(a) = z_a a$ for all $z \in H$;

(B) if for all $n \geq 1$ and $x_1, \ldots, x_n \in \mathbb{A}$ there exists a finite dimensional C^*-algebra $B \subseteq \mathbb{A}$ such that $x_1, \ldots, x_n \in B$;

(C') if for all $w \in W \backslash W_0$, $e \in \mathbb{P}$ there exists $p \in \mathbb{P}$ such that $p \leq e$ and $pwp = 0$; and

(C'*) if there exists a subset $\mathbb{P}_2 \subseteq \mathbb{P}$ such that for each $q \in \mathbb{P}$ there exists a $\rho \in \mathbb{P}_2$ such that $\rho \preceq q$ in $\mathbb{A}$ and for all $w \in W \backslash W_0, e \in \mathbb{P}_2$ there exists $p \in \mathbb{P}_2$ such that $p \leq e$ and $pwp = 0$.

Lemma 2.7. The system $(\mathbb{F}, \mathbb{I}, H)$ as defined above satisfies property (A).

Proof. Let $z \in H$ and $x \in \mathbb{I}$. Then, it is easy to see that $\pi\Gamma_z = \gamma_{\theta^{-1}(z)}\pi$ for all $z \in \mathbb{T}^{\mathcal{A}}$. Thus $\pi(\Gamma_z(x)) = \gamma_{\theta^{-1}(z)}(\pi(x)) = 0$ so that $\Gamma_z(x) \in \mathbb{I}$. □

Since $(\mathbb{F}, \mathbb{I}, H)$ satisfies (A), by [2, Lemma 3.1] there exists a *balance function* $\mathrm{bal} : W\backslash\{0\} \longrightarrow \hat{H}$ satisfying

$$\mathrm{bal}(xy) = \mathrm{bal}(x)\,\mathrm{bal}(y), \text{ and } \mathrm{bal}(z^*) = \mathrm{bal}(z)^{-1}$$

for all $x, y, z \in W$ such that $xy \neq 0$ and $z \neq 0$. It is easy to see that under the canonical identification of $\hat{H} \cong \hat{\mathbb{T}}^k$ with $\mathbb{Z}^k$ we have $\mathrm{bal}(t_\lambda) = d(\lambda)$.[4]

For $\lambda \in \Lambda$ define $t_\lambda := \Psi^{-1}(s_{\mathcal{E}}(\lambda))$ and note that this definition agrees with the original definition of t_λ when $\lambda \in \Sigma \cup \Lambda^0$. By construction $\{t_\lambda \mid \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in $\mathbb{F}/\mathbb{I}$.

Lemma 2.8 (Definition)**.** Let (Λ, d) be a finitely aligned k-graph. Suppose that $\{\tau_\lambda \mid \lambda \in \Lambda\}$ is a family of partial isometries satisfying (TCK1)–(TCK3) in a *-algebra B. Given a finite subset $E \subseteq \Lambda$, there exists a finite subset ΠE such that $E \subseteq \Pi E$ and

$$M^\tau_{\Pi E} := \mathrm{span}\{\tau_\lambda \tau^*_\mu \mid \lambda, \mu \in \Pi E,\ s(\lambda) = s(\mu),\ d(\lambda) = d(\mu)\}$$

is a finite dimensional *-subalgebra of B. Moreover, if E and F are finite subsets of Λ then $\Pi E \subseteq \Pi F$ so that $M^\tau_{\Pi E} \subseteq M^\tau_{\Pi F}$.

Proof. The assertions are essentially collected from [16, Lemma 3.4.2, Lemma 3.4.7]. □

Lemma 2.9. Let (Λ, d) and $(\mathbb{F}, \mathbb{I}, H)$ be as above. Then

$$\begin{aligned} \mathbb{A} &= \mathrm{span}\{t_\lambda t^*_\mu \mid \lambda, \mu \in \Lambda,\ s(\lambda) = s(\mu),\ d(\lambda) = d(\mu)\} \\ &= \bigcup_{\substack{E \subseteq \Lambda \\ \text{finite}}} M^t_{\Pi E}. \end{aligned}$$

Hence $(\mathbb{F}, \mathbb{I}, H)$ satisfies property (B).

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[4]In more detail we have $c_{\tilde{t}_\lambda} \circ \theta = \chi_{d(\lambda)}$ where $c_{\tilde{t}_\lambda}$ is the character of H defined in [4] and for each $n \in \mathbb{Z}^k$, χ_n is the character $z \mapsto z^n$ for all $z \in \mathbb{T}^k$.

Proof. The first equality follows from the identification $\mathrm{bal}(t_\lambda) = d(\lambda)$ for all $\lambda \in \Sigma \cup \Lambda^0$ and the relations (TCK1)–(TCK3). The second equality follows from Lemma 2.8 and the final assertion is obvious. $\square$

Lemma 2.10 ([16, Proposition 3.5.3])**.** Let (Λ, d) be a finitely-aligned k-graph, let $\{\tau_\lambda \mid \lambda \in \Lambda\}$ be a family of partial isometries satisfying (TCK1)–(TCK3) in a *-algebra B, and let $E \subseteq \Lambda$ be finite. Define

$$\Theta^{\Pi E}_{\lambda,\mu}(\tau) := \tau_\lambda \tau^*_\lambda \prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (\tau_\lambda \tau^*_\lambda - \tau_{\lambda\nu}\tau^*_{\lambda\nu}).$$

Then $\{\Theta^{\Pi E}_{\lambda,\mu}(\tau) \mid \lambda, \mu \in \Lambda,\ s(\lambda) = s(\mu),\ d(\lambda) = d(\mu)\}$ is a collection of matrix units for $M^\tau_{\Pi E}$.

Lemma 2.11. If Λ satisfies **(AP)**, then the system $(\mathbb{F}, \mathbb{I}, H)$ as defined above satisfies condition (C'*).

Proof. We claim that $\{t_\lambda t_\lambda \mid \lambda \in \Lambda\}$ is a valid candidate for $\mathbb{P}_2$. Indeed, if $q \in \mathbb{P}$ then $q \in M^t_{\Pi E}$ for some finite subset $E \subseteq \Lambda$ by Lemma 2.9. By [16, Lemma 3.6.2] there exists a non-zero projection q' such that $q' \leq q$ and $q' \in M^t_{\Pi E}(n, v)$ for some $v \in s(\Pi E)$ and some $n \in d(\Pi E v)$, where $M^t_{\Pi E}(n, v)$ is a simple finite dimensional *-algebra spanned by the family of (non-zero) matrix units $\{\Theta^{\Pi E}_{\lambda,\mu}(t) \mid \lambda, \mu \in \Pi E v \cap \Lambda^n\}$. Therefore, there exists a partial isometry $s \in M^t_{\Pi E}(n, v)$ such that $s^*s = q'$ and $ss^* = \sum_{\lambda \in F} \Theta^{\Pi E}_{\lambda,\lambda}(t)$ for some finite subset $F \subseteq \Pi E v \cap \Lambda^n$. Now $t_\lambda t^*_\lambda \leq \Theta^{\Pi E}_{\lambda,\lambda}(t)$ for all $\lambda \in \Lambda$ therefore, choose any $\lambda_0 \in F$ and set $\rho := t_{\lambda_0} t^*_{\lambda_0}$. Then $\rho \in \mathbb{P}_2$ and $\sigma := s^*\rho$ implements the relation $\rho \preceq q$ in $\mathbb{A}$ as required.

If $w \in W$ then a simple inductive argument using (TCK1)–(TCK3) shows that $w = \sum_{(\lambda,\mu) \in F} t_\lambda t^*_\mu$ for some finite subset $F \subseteq \{(\xi, \eta) \in \Lambda \times \Lambda \mid d(\xi) = m,\ d(\eta) = n,\ s(\xi) = s(\eta)\}$ for some $m, n \in \mathbb{N}^k$. Furthermore, if $w \notin W_0$ then we must have $m \neq n$. We shall prove that given any $w = \sum_{(\xi,\eta) \in F} t_\xi t^*_\eta \in W \backslash W_0$ and any $e \in \mathbb{P}_2$ there exists $p \in \mathbb{P}_2$ such that $pwp = 0$ by induction on the cardinality of F.

Suppose that $|F| = 1$, then $w = t_\xi t^*_\eta$ for some $\xi, \eta \in \Lambda$ with $s(\xi) = s(\eta)$ and $d(\xi) \neq d(\eta)$. We also have $e = t_{\lambda_0} t^*_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Set $N := d(\xi) \vee d(\eta) \vee d(\lambda_0)$. There are two cases to consider.

Case 1. Suppose that $\mathrm{MCE}(\{\lambda_0, \xi, \eta\}) = \emptyset$ and let $\lambda \in \lambda_0 \Lambda^{\leq N - d(\lambda_0)}$. If $d(\lambda) = N$ then either $\lambda(0, d(\xi)) \neq \xi$ or $\lambda(0, d(\eta)) \neq \eta$. In either case we have

$$\begin{aligned} t_\lambda t^*_\lambda t_\xi t^*_\eta t_\lambda t^*_\lambda &= t_\lambda t^*_{\lambda(d(\xi),N)} t^*_{\lambda(0,d(\xi))} t_\xi t^*_\eta t_{\lambda(0,d(\eta))} t_{\lambda(d(\eta),N)} \\ &= 0. \end{aligned}$$

On the other hand, if $d(\lambda) < N$, then there exists $1 \leq i \leq k$ such that $d(\lambda)_i < d(\xi)_i$ or $d(\lambda)_i < d(\eta)_i$. Without loss of generality, suppose $d(\lambda)_i < d(\xi)_i$. Then $\Lambda^{\min}(\lambda, \xi) = \emptyset$, otherwise there exists $\mu \in \mathrm{MCE}(\lambda, \xi)$ so that $\mu(d(\lambda), d(\lambda) + e_i) \in s(\lambda)\Lambda^{e_i}$ contradicting the fact that $\Lambda^{e_i} = \emptyset$. Therefore,

$$t_\lambda t^*_\lambda t_\xi t^*_\eta t_\lambda t_\lambda = 0.$$

Case 2. Suppose that $\mathrm{MCE}(\{\lambda_0, \xi, \eta\}) \neq \emptyset$. Then choose any $\tilde{\lambda} \in \mathrm{MCE}(\{\lambda_0, \xi, \eta\})$ and set $\mu := \tilde{\lambda}(d(\xi), N)$, $\nu := \tilde{\lambda}(d(\eta), N)$ and $v := s(\tilde{\lambda})$. Since Λ satisfies **(AP)** there exists $x \in v\Lambda^*$ satisfying (1) in Definition 2.4. By [16, Lemma 4.5.3], there exists an $n \in \mathbb{N}^k$ such that $n \leq d(x)$ and $\Lambda^{\min}(\mu x(0, n), \nu x(0, n)) = \emptyset$. Set $L := N + n$. Now $L \leq N + d(x)$, therefore we may define $p := t_\lambda t^*_\lambda \in \mathbb{P}_2$ where $\lambda := (\tilde{\lambda} x)(0, L)$. We have $\lambda(0, d(\lambda_0)) = \lambda_0$, $\lambda(0, d(\xi)) = \xi$ and $\lambda(0, d(\eta)) = \eta$, so that $p \leq t_{\lambda_0} t^*_{\lambda_0} = e$ and

$$\begin{aligned} pwp &= t_\lambda t^*_{\lambda(d(\xi),L)} t_{\lambda(d(\eta),L)} t^*_\lambda \\ &= 0 \end{aligned}$$

since $\lambda(d(\xi), L) = \mu x(0, n)$, $\lambda(d(\eta), L) = \nu x(0, n)$ and $\Lambda^{\min}(\mu x(0, n), \nu x(0, n)) = \emptyset$. We have now proved the first step of the induction process.

For the induction hypothesis, assume that given any $w \in W$ that can be expressed as $w = \sum_{(\xi,\eta)\in F} t_\xi t_\eta^*$ for some subset $F \subseteq \Lambda^n \times \Lambda^m$ $(n \neq m)$ of cardinality j, and given any $e \in \mathbb{P}_2$, there exists $p \in \mathbb{P}_2$ such that $pwp = 0$.

Now given any $w \in W$ with $w = \sum_{(\xi,\eta)\in F} t_\xi t_\eta^*$ with $F \subseteq \Lambda^n \times \Lambda^m$, $|F| = j + 1$ and any $e \in \mathbb{P}_2$, choose any $(\xi_0, \eta_0) \in F$ and set $w' = \sum_{(\xi,\eta)\in F\setminus\{(\xi_0,\eta_0)\}} t_\xi t_\eta^*$. Then there exists a $p_0 \in \mathbb{P}_2$ such that $p_0 \leq e$ and $p_0 w' p_0 = 0$. Moreover, by the above, there exists $p \in \mathbb{P}_2$ such that $p \leq p_0$ and $p t_{\xi_0} t_{\eta_0}^* p = 0$. Therefore $p \leq e$ and $pwp = pp_0 w' p_0 p + p t_{\xi_0} t_{\eta_0}^* p = 0$, as required. □

Proposition 2.12 (cf. [16, Proposition 3.5.8])**.** Let (Λ, d) be a finitely aligned k-graph and let $\{\tau_\lambda \mid \lambda \in \Lambda\}$, $\{\tau'_\lambda \mid \lambda \in \Lambda\}$ be two families of partial isometries in *-algebras A, B respectively, satisfying (TCK1)–(TCK3). Furthermore, suppose that $\tau_v \neq 0$ and $\tau'_v \neq 0$ for all $v \in \Lambda^0$. Then there exists a *-isomorphism

$$\pi_{\tau,\tau'} : \operatorname{span}\{\tau_\lambda \tau_\mu^* \mid d(\lambda) = d(\mu)\} \longrightarrow \operatorname{span}\{\tau'_\lambda {\tau'}_\mu^* \mid d(\lambda) = d(\mu)\}$$

if and only if for all $E \in \mathrm{FE}(\Lambda)$ we have

$$\prod_{\lambda\in E} (\tau_{r(E)} - \tau_\lambda \tau_\lambda^*) = 0 \iff \prod_{\lambda\in E} (\tau'_{r(E)} - \tau'_\lambda {\tau'}_\lambda^*) = 0.$$

Proof. This is proved in the proof of [16, Proposition 3.5.8], the only difference being that we are content with considering *-algebras rather than C^*-algebras. □

Lemma 2.13 ([16, Corollary 4.3.10])**.** Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$. Let $\{s_\mathcal{E}(\lambda) \mid \lambda \in \Lambda\}$ be the universal generating Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in $C^*(\Lambda; \mathcal{E})$. Then

(1) $s_\mathcal{E}(v) \neq 0$ for all $v \in \Lambda^0$; and
(2) if $E \in \mathrm{FE}(\Lambda)$ then $\prod_{\lambda\in E}(s_\mathcal{E}(r(E)) - s_\mathcal{E}(\lambda)s_\mathcal{E}(\lambda)^*) = 0$ if and only if E belongs to $\bar{\mathcal{E}}$.

Theorem 2.14. Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Suppose that Λ satisfies **(AP)**. Let $\{\tau_\lambda \mid \lambda \in \Lambda\}$ be a Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in a C^*-algebra B such that

(i) $\tau_v \neq 0$ for all $v \in \Lambda^0$, and
(ii) $\prod_{\lambda\in F}(\tau_{r(F)} - \tau_\lambda \tau_\lambda^*) \neq 0$ for all $F \in \mathrm{FE}(\Lambda)\setminus\bar{\mathcal{E}}$.

Then the *-homomorphism $\pi_\tau^\mathcal{E} : C^*(\Lambda; \mathcal{E}) \longrightarrow B$ satisfying $\pi_\tau^\mathcal{E}(s_\mathcal{E}(\lambda)) = \tau_\lambda$ for all $\lambda \in \Lambda$ is faithful.

Proof. Define $\rho := \pi_\tau^\mathcal{E}\Psi$. Now $\pi_\tau^\mathcal{E}$ is injective on $\Psi(\mathbb{A})$ by Lemma 2.8, Lemma 2.13 and Proposition 2.12. Moreover, Ψ is injective on $\mathbb{A}$ since Ψ is known to be injective, thus ρ is also injective on $\mathbb{A}$. By Lemmas 2.7 and 2.9 $(\mathbb{F}, \mathbb{I}, H)$ satisfies (A) and (B). Furthermore, as $(\mathbb{F}, \mathbb{I}, H)$ satisfies property (C'*), by Lemma 2.11, it must also satisfy property (C'), by [4, Lemma 2.4]. We may now apply [4, Theorem 2.3] to conclude that $\pi_\tau^\mathcal{E}$ is injective □

Remark 2.15. It seems that we have proved an uniqueness theorem for a wider class of relative higher-rank graph C^*-algebras than has been done previously. Indeed, by Remark 2.5 we may deduce [16, Theorem 4.5.2],[16, Theorem 4.6.5], [13, Theorem 4.5] from Theorem 2.14. Compare also with [10, Theorem 7.1], [10, Remarks 7.3], [16, Remark 4.6.7] and [2, Proposition 4.3].

Remark 2.16. It is likely that Theorem 2.14 also holds for uncountable finitely aligned higher-rank graphs.

3. A representation of higher rank Cuntz-Krieger type C^*-algebras as higher rank graph C^*-algebras

In [3] the first named author introduced a class of higher rank Cuntz-Krieger type C^*-algebras, which were shown to be of Cuntz-Krieger type. A higher rank Cuntz-Krieger type C^*-algebra is given by a set $\mathcal{A}$ (the alphabet), which is endowed with a fixed partition $V := \{V_i \mid; i \in I\}$, a self-adjoint, two-sided idea $\mathbb{I}$ in the free *-algebra $\mathbb{F}$ generetated by $\mathcal{A}$, which satisfies certain properties described below, and a family of matrices $\{A_i \mid i \in I\}$ with $A_i \in \{0,1\}^{V_i \times V_i}$. We shall only be concerned with the case when I and V_i, $(i \in I)$ are finite.

For each subset X of $\mathbb{F}$ we let $X^* := \{x^* \mid x \in X\}$ and put $X^{\circledast} := X \cup X^*$. For ease of notation we define two relations on $\mathcal{A}^{\circledast}$:

$$\begin{aligned} a \sim b \quad &\iff \quad \exists\, i \in I \text{ such that } a, b \in V_i \text{ or } a, b \in V^{\circledast}, \\ a \| b \quad &\iff \quad \exists\, i, j \in I \text{ such that } i \neq j \text{ and } a \in V_i^{\circledast}, \text{ and } b \in V_j^{\circledast}. \end{aligned}$$

For each element $x \in \mathbb{F}$ we let $\tilde{x}$ be the image of x in $\mathbb{F}/\mathbb{I}$ under the natural *-homomorphism. Similarly we let $\tilde{X}$ be the image of a subset X. For each $a \in \mathcal{A}$ let $Q_a := a^*a$, $P_a := aa^*$, $q_a := \tilde{Q_a}$ and $p_a := \tilde{P_a}$. As stated above, we assume the ideal $\mathbb{I}$ satisfies some properties, which are (cf. [3, Definition 2.1]):[5]

Cuntz-Krieger relations. For each $i \in I$ and $a \in V_i$ the following relations hold in $\mathbb{F}/\mathbb{I}$:

$$a = aa^*a, \ q_a q_b = q_b q_a, \ p_a p_b = \delta_{a,b} p_a, \ q_a p_b = A_i(a,b) p_b.$$

Permutation rules. For all $a, b \in \mathcal{A}^{\circledast}$ such that $a \| b$ we have $ab = 0$ or there exist $A, B \in \mathcal{A}^{\circledast}$ such that $A \sim a$, $B \sim b$ and

$$ab = BA \text{ and } Ab^* = B^*a.$$

Invariance under the gauge actions. The ideal $\mathbb{I}$ is invariant under the *-automorphisms $\phi_z : \mathbb{F} \longrightarrow \mathbb{F}$ given by $\phi_z(a) = z_a a$ for all $a \in \mathcal{A}$ and $z = \{z_a\}_{a \in \mathcal{A}} \in H$, where

$$H := \{\{z_a\}_{a \in \mathcal{A}} \in \mathbb{T}^{\mathcal{A}} \mid \forall\, a, b \in \mathcal{A}, \ a \sim b \implies z_a = z_b\}.$$

Projections property. For all $x = x_1 \cdots x_m$ such that $x_j \in \tilde{\mathcal{A}}$ with $xx^* \neq 0$, and sequences $\{a_n\}_{n \geq 1} \subset V_i$ $(i \in I)$, there exists $N \geq 1$ such that $xx^* a_1 \cdots a_N a_N^* \cdots a_1^* \neq xx^*$.

Saturating $\mathbb{A}_{00}$-faithful representation. There exists a representation $\pi : \mathbb{F}/\mathbb{I} \longrightarrow B(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space) such that for all $i \in I$ the strong operator sum $U_i := \sum_{b \in V_i} \pi(p_b)$ satisfies $U_i \pi(a) = \pi(a) U_i = \pi(a)$ for all $a \in \tilde{V_i}$. Such a representation is called *saturating.* Moreover we assume that this π is faithful on the *-subalgebra $\mathbb{A}_{00}$ in $\mathbb{F}/\mathbb{I}$ generated by $\{aa^* \in \mathbb{F}/\mathbb{I} \mid a \in \tilde{\mathcal{A}}\}$.

Let $W := W(\mathcal{A})$ be the image in $\mathbb{F}/\mathbb{I}$ of the set of all words in $\mathbb{F}$ consisting of letters in $\mathcal{A}$. The existence of a balance function $\mathrm{bal} : W \backslash \{0\} \longrightarrow \bigoplus_{i \in I} \mathbb{Z}$, which has the form $\mathrm{bal}(a) = \delta_i$ for $a \in \tilde{V_i}$ is shown in [3, §3].

We can now recall the definition of a higher rank Cuntz-Krieger algebra (in the case when V is a finite partition consisting of finite sets).

Definition 3.1 ([3, Definition 2.2])**.** Let $(\mathcal{A}, V, \{A_i \mid i \in I\}, \mathbb{I})$ be as above. Let $\pi_1 : \mathbb{F}/\mathbb{I} \longrightarrow A_1$ be a *-homomorphism into a C^*-algebra A_1 such that π_1 has dense image and is faithful on $\mathbb{A}_{00}$. Then we call the C^*-algebra A_1 a *higher rank Cuntz-Krieger type algebra* and denote it by $\mathcal{O}_{\mathbb{F},\mathbb{I}}$.

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[5] As we are assuming that $\mathcal{A}$ has a finite partition consisting of finite sets, the finiteness property in [3, Definition 2.1] is satisfied automatically (see [3, Remark 2.5]).

Remark 3.2. The definition of $\mathcal{O}_{\mathbb{F},\mathbb{I}}$ does not depend on the representation π_1 up to *-isomorphism due to the uniqueness theorem [3, Theorem 2.3].

Let $\mathrm{Obj}(\Lambda) := \{(a_1, \ldots, a_k) \in V_1 \times \cdots \times V_k \mid a_1 a_1^* \cdots a_k a_k^* \neq 0\}$. For $a \in \mathrm{Obj}(\Lambda)$ let $s_a = a_1 a_1^* \cdots a_k a_k^*$. We define

$$\mathrm{Mor}(\Lambda) := \{s_a w s_b \mid a, b \in \mathrm{Obj}(\Lambda), w \in W \; s_a w s_b \neq 0\} \cup \{s_a \mid a \in \mathrm{Obj}(\Lambda)\}.$$

We define the range and source map as follows:

$$r(s_a w s_b) = r(s_a) = s_a \qquad s(s_a w s_b) = s(s_b) = s_b,$$

for each $a, b \in \mathrm{Obj}(\Lambda)$ and $w \in W$. Composition in Λ is given by multiplication in $\mathbb{F}/\mathbb{I}$. The following lemma shows that the composition is well-defined. Furthermore it is clear that the set of identity morphisms is $\{s_a \mid a \in \mathrm{Obj}(\Lambda)\}$ and that the composition is associative.

Lemma 3.3. If $a, b \in \mathrm{Obj}(\Lambda)$ and $w \in W$ such that $s_a w s_b \neq 0$ then $s_a w s_b = w s_b$ and a is uniquely determined by $w s_b$.

Proof. Let $a = (a_1, \ldots, a_k), b = (b_1, \ldots, b_k) \in \mathrm{Obj}(\Lambda)$ and $w \in W$ such that $s_a w s_b \neq 0$. We shall prove the Lemma by induction on the length[6] of the word w So let w be of length 1, i.e. $w \in \tilde{\mathcal{A}}$, and without loss of generality, suppose that $w \in V_k$. By [3, Lemma 4.3] we have $b_i b_i^* b_j b_j^* = b_j b_j^* b_i b_i^*$ for all $i, j \in \{1, \ldots, k\}$, therefore $w b_j \neq 0$ for all $j = 1, \ldots, k$. Thus by multiple applications of [3, Lemma 4.1] there exist $(c_1, \ldots, c_{k-1}) \in \tilde{V_1} \times \cdots \times \tilde{V}_{k-1}$ such that $w s_b = c_1 c_1^* \cdots c_{k-1} c_{k-1} c_k c_k^* w s$ where $c_k := w \in \tilde{V_k}$. Therefore we have shown that $w s_b = s_c w s_b$ where $c = (c_1, \ldots, c_k) \in \mathrm{Obj}(\Lambda)$ and it remains to show that $c = a$. To this end, note that for all $x, y \in \mathrm{Obj}(\Lambda)$ we have $s_x s_y = \delta_{x,y} s_x$ by [3, Lemma 4.3] and the Cuntz-Krieger relations $p_{x_i} p_{y_i} = \delta_{x_i, y_i} p_{x_i}$ for all $i = 1, \ldots, k$. Therefore since $0 \neq s_a w s_b = s_a s_c w s_b$ we have $s_a = s_c$. □

The degree functor $d : \Lambda \longrightarrow \mathbb{N}^k$ is given by the balance function, i.e. $d(\lambda) = \mathrm{bal}(\lambda)$ for all $\lambda \in \Lambda$.

Lemma 3.4. Let Λ be the category with degree functor $d : \Lambda \longrightarrow \mathbb{N}^k$ as defined above. Then d satisfies the factorisation property.

Proof. We must show that given any $\lambda \in \Lambda$ such that $d(\lambda) = m + n$ then there exist unique $\xi, \eta \in \Lambda$ such that $\lambda = \xi\eta$. The existence of such a decomposition follows from an inductive argument on the length of the word w, which uses the permutation rules in $\mathbb{F}/\mathbb{I}$ (it is trivial if $d(\lambda) = 0$).

The uniqueness of the decomposition follows from [3, Lemma 4.4]. □

Theorem 3.5. Let Λ and $d : \Lambda \longrightarrow \mathbb{N}^k$ be the category and functor respectively defined above. Then Λ is a row-finite k-graph with no sources.

Proof. Lemma 3.4 shows that (Λ, d) is a k-graph. That Λ is row-finite follows from the facts that each morphism of Λ is uniquely determined by its range, source and degree (by Lemma 3.3), and that $\mathrm{Obj}(\Lambda)$ is finite. That Λ has no sources follows from the existence of a $\mathbb{A}_{00}$-faithful saturating representation: given any $a = (a_1, \ldots, a_k) \in \mathrm{Obj}(\Lambda)$ and $i = 1, \ldots, k$, we have

$$\begin{aligned} 0 \neq a_1 a_1^* \cdots a_k a_k^* a_i &= a_1 a_1^* \cdots a_k a_k^* a_i b_1 b_1^* \cdots b_{i-1} b_{i-1}^* b_{i+1} b_{i+1}^* b_k b_k^* \\ &= \sum_{b_i \in V_i} a_1 a_1^* \cdots a_k a_k^* a_i b_1 b_1^* \cdots b_k b_k^* \end{aligned}$$

[6]Suppose that $0 \neq w = w_1 \cdots w_n = x_1 \cdots x_m$ for some $m, n \geq 1$ and $w_i, x_j \in \tilde{\mathcal{A}}$ for $i = 1, \ldots, n,\ j = 1, \ldots, m$. Then an application of the balance function to w shows that $m = n$. Therefore the concept of a word consisting of letters in $\tilde{\mathcal{A}}$ having a (unique) length is well-defined.

for some $(b_1,\dots,b_{i-1},b_{i+1},\dots,b_k)\in V_1\times\cdots V_{i-1}\times V_{i+1}\cdots V_k$ (uniquely determined by a and a_i). Thus, there exists a $b_i\in V_i$ such that $a_1a_1^*\cdots a_ka_k^*a_ib_1b_1^*\cdots b_kb_k^*\neq 0$ therefore $a\Lambda^{e_i}\neq\emptyset$. □

Lemma 3.6. Let Λ be the k-graph defined above. For each $\lambda\in\Lambda$ let $t_\lambda=\pi_1(\lambda)\in\mathcal{O}_{\mathbb{F},\mathbb{I}}$, where π_1 is the representation in Definition 3.1 . Then the set $\{t_\lambda\mid\lambda\in\Lambda\}$ is *-representation of Λ in $\mathcal{O}_{\mathbb{F},\mathbb{I}}$ in the sense of [11, Definitions 1.5].

Proof. First note that it is clear that $\{t_\lambda\mid\lambda\in\Lambda\}$ is a set of partial isometries. The set $\{t_v\mid v\in\Lambda^0\}$ is clearly a set of projections, which are are mutually orthogonal by the Cuntz-Krieger relations and commutativity of the range projections [3, Lemma 4.3]. Thus we have shown that [11, Definitions 1.5, (i)] holds. Lemma 3.3 ensures that [11, Definitions 1.5, (ii)] holds. An application of [3, Lemma 4.3] and the Cuntz-Krieger relations shows that [11, Definitions 1.5, (iii)] holds. To prove that [11, Definitions 1.5, (iv)] holds, i.e that $t_v=\sum_{\lambda\in v\Lambda^n}t_\lambda t_\lambda^*$ for all $n\in\mathbb{N}^k$ we make use of the fact that we need only prove that the relation holds for $n=e_i$ for all $i=1,\dots,k$ (cf. [11, Remarks 1.6, (iii)]). To this end we note that the existence of a saturating $\mathbb{A}_{00}$-faithful representation ensures that $u_i:=\pi_1(\sum_{a\in V_i}p_a)$ is a unit for $\pi_1(b)$ for all $b\in V_i$, thus for $v=(v_1,\dots,v_k)\in\mathrm{Obj}(\Lambda)$ we have

$$\begin{aligned} v\Lambda^{e_i} &= \{s_vv_is_b\mid b\in\mathrm{Obj}(\Lambda),\ A_i(v_i,b_i)=1 \text{ and for } j\neq i,\\ &\qquad b_j \text{ is uniquely determined by } v_i \text{ and } v_j\}, \end{aligned}$$

and

$$\begin{aligned} \sum_{\lambda\in v\Lambda^{e_i}}t_\lambda t_\lambda^* &= \pi_i\left(\sum_{b_i\in V_i}v_is_bv_i^*\right)\\ &= \pi_1(v_i)\sum_{b_i\in V_i}\pi_1(p_{b_i})\pi_1(p_{b_1}\cdots p_{b_{i-1}}p_{i+1}\cdots p_{b_k}v_i^*)\\ &= \pi_1(v_ip_{b_1}\cdots p_{b_{i-1}}p_{i+1}\cdots p_{b_k}v_i^*)\\ &= \pi_1(s_v)=t_v. \end{aligned}$$

□

We now state and prove the main result of this section.

Theorem 3.7. Let $(\mathcal{A},V,\{A_i\mid i\in I\},\mathbb{I})$ and Λ be as described in this section. Then

(1) $C^*(\Lambda)$ is canonically *-isomorphic to a *-subalgebra of $\mathcal{O}_{\mathbb{F},\mathbb{I}}$; and
(2) if we make the further assumption that the following holds in $\mathcal{O}_{\mathbb{F},\mathbb{I}}$

$$\sum_{b_1\in\tilde{V_1},\dots,b_k\in\tilde{V_k}}\pi_1(b_1b_1^*\cdots b_kb_k^*)=1,$$

then $\mathcal{O}_{\mathbb{F},\mathbb{I}}\cong C^*(\Lambda)$.

Proof. To prove (1), note that we have already shown in Lemma 3.6 that $\{t_\lambda\mid\lambda\in\Lambda\}$ is a *-representation of Λ in $\mathcal{O}_{\mathbb{F},\mathbb{I}}$, therefore by universality of the k-graph C^*-algebras there exists a *-homomorphism $\rho:C^*(\Lambda)\longrightarrow\mathcal{O}_{\mathbb{F},\mathbb{I}}$. Now $t_v\neq 0$ for all $v\in\mathrm{Obj}(\Lambda)$ by definition and the invariance under the gauge automorphisms property ensures the existence of a gauge action on $\mathcal{O}_{\mathbb{F},\mathbb{I}}$ that intertwines ρ and the canonical gauge action on $C^*(\Lambda)$. Thus by gauge invariant uniqueness theorem [11, Theorem 3.4] ρ is injective.

To prove (2), note that in addition we have for all $i=1,\dots,k$ and $a\in\tilde{V_i}$:

$$\pi_1(a)=\sum_{b\in\mathrm{Obj}(\Lambda)}\pi_1(as_b).$$

Therefore $\{t_\lambda \mid \lambda \in \Lambda\}$ generates $\mathcal{O}_{\mathbb{F},\mathbb{I}}$ and ρ is surjective. □

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