Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbst-ständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Kurzfassung der Dissertation

(1) Wir geben eine Variante der Doob-Meyer-Zerlegung aus der Wahrscheinlichkeitstheorie. Genauer zerlegen wir Funktionen mit Werten in einem Hilbertraum und wo der Definitionsbereich eine total geordnete Menge ist. Wir geben zwei Varianten für die Zerlegung.

Die erste ("diskrete") Zerlegung zerlegt eine Klasse von nicht notwendigerweise rechtsstetigen Prozessen. Der Beweis is überraschend kurz und elementar. Die zweite ("integrierbare") Zerlegung zerlegt eine Klasse von rechtsstetigen Prozessen. Die Zeitskala ist hier mit einem endlichen Maß ausgestattet, welches in der Zerlegung berücksichtigt wird.

(2) Betrachte einen kompakten Hausdorff-Raum K und einen Zustand $\varphi \in C(K)^*$. Sei $X \subseteq C(K)$ ein linearer Teilraum. Des Weiteren seien $c \leq 1 \leq C$ reelle Konstanten. Dann wird X durch ein Funktional $\psi \in C(K)^*$ mit $\psi(\mathbf{1}) = 1$ und $c\varphi \leq \psi \leq C\varphi$ annihiliert, genau dann wenn

$$\forall x \in X : \varphi(x) \leq \frac{1}{C} \min (C-1)a + (c-1)b$$

wo das Minimum über alle reellen Zahlen a, b mit

$$\varphi(x) \leq a \leq \max(s),\, b \leq 0,\, \varphi((x-a) \vee b) = 0$$

genommen wird. Der Fall $C(K) \cong L^\infty(\mathbb{P})$, wo $\mathbb{P}$ ein Wahrscheinlichkeitsmaß ist, findet vielleicht Anwendung in der Finanzmathematik.

Summary of the thesis

(1) We give a variant of the Doob-Meyer-decomposition from probability theory. More precisely we Doob-Meyer-decompose functions with domain a

totally ordered set (time-scale) and image in a Hilbert space. We give two variants for the composition.

The first ("discrete") decomposition decomposes a class of not necessarily right-continuous processes. The proof is surprisingly short and elementary. The second ("integrable") decomposition decomposes a class of right-continuous processes. The time scale is here endowed with a finite measure which is taken into account in the decomposition.

(2) Consider a compact Hausfdorff space K and a state $\varphi \in C(K)^*$. Let $X \subseteq C(K)$ be a linear subspace. Furthermore let $c \leq 1 \leq C$ be real constants. Then X is annihilated by a functional $\psi \in C(K)^*$ with $\psi(\mathbf{1}) = 1$ and $c\varphi \leq \psi \leq C\varphi$ if and only if

$$\forall x \in X : \varphi(x) \leq \frac{1}{C} \min\,(C-1)a + (c-1)b$$

where the minimum is taken over all reals a, b with

$$\varphi(x) \leq a \leq \max(s),\ b \leq 0,\ \varphi((x-a) \vee b) = 0.$$

The case $C(K) \cong L^\infty(\mathbb{P})$, where $\mathbb{P}$ is a probability measure, may have application in financial mathematics.

Contents

Chapter 1

Introduction

1.1 Doob-Meyer-decomposition

The first part of this thesis deals with the Doob-Meyer-decomposition, which is well-known to probabilists. We give a new proof of the Doob-Meyer-decomposition for square integrable stochastic processes. Actually we replace here the Hilbert space $L^2(\Omega, \mathcal{F}, P)$ by an arbitrary Hilbert space $\mathcal{H}$ and the common time scale $\mathbb{R}_+$ by an arbitrary totally ordered set $\mathcal{R}$.

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a countable **filtration** $(\mathcal{F}_n)_{n\in\mathbb{N}}$, i.e. an **increasing** family of σ-algebras $\mathcal{F}_n \subseteq \mathcal{F}$ $(n \in \mathbb{N})$,

$$\mathcal{F}_k \subseteq \mathcal{F}_n \text{ for all } k \leq n \in \mathbb{N}.$$

The interpretation is that each $n \in \mathbb{N}$ represents a point of time. And the σ-algebra $\mathcal{F}_n$ is the collection of all possible events of a probabilistic process at the point of time n. When time goes on, the set $\mathcal{F}_n$ of possible outgoings will grow. That is why the filtration is increasing.

Now let us given an adapted **stochastic process**

$$X : \mathbb{N} \to L^1(\Omega, \mathcal{F}, P)$$

with respect to our filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, where per definition **adaptivity** means

$$X_n \in L^1(\Omega, \mathcal{F}_n, P) \text{ for all } n \in \mathbb{N}.$$

Note here that

$$L^1(\Omega, \mathcal{F}_n, P) \subseteq L^1(\Omega, \mathcal{F}, P) \text{ for all } n \in \mathbb{N}.$$

Then the **Doob-decomposition** states the following: X can be decomposed into two stochastic processes

$$X = M + A,$$

where $M : \mathbb{N} \to L^1(\Omega, \mathcal{F}, P)$ is a martingale and where $A : \mathbb{N} \to L^1(\Omega, \mathcal{F}, P)$ is predictable. **Predictability** means that

$$A_n \in L^1(\Omega, \mathcal{F}_{n-1}, P) \text{ for all } n \in \mathbb{N}.$$

So predictability is somewhat stronger than adaptivity. A predictable process is even adapted to a filtration which runs one step after the given filtration.

A **martingale** M fulfills per def.

$$E(M_n|\mathcal{F}_k) = M_k \quad \text{for all } k \leq n \in \mathbb{N}.$$

The interpretation of a martingale M is that: If M codes a game of chance, then this game is a fair game. On the long turn you will neither win nor loose money.

$E(.|\mathcal{F}_n) \in L^1(\Omega, \mathcal{F}, P)$ denotes the **expectation with respect to the σ-algebra** $\mathcal{F}_n$. Probabilistically spoken, for a random variable $X \in L^1(\Omega, \mathcal{F}, P)$, $E(X|\mathcal{F}_n)$ is the $\mathcal{F}_n$-measurable and integrable random variable which gives the expectation of X with respect to the information given to the point of time n.

Geometrically spoken, for a given vector $X \in L^1(\Omega, \mathcal{F}, P)$, $E(X|\mathcal{F}_n)$ is the vector in $L^1(\Omega, \mathcal{F}_n, P)$ which comes out if we project X onto the subspace $L^1(\Omega, \mathcal{F}_n, P)$. Since in general there are infinitely many possibilities how to project onto a subspace, the projection $E(.|\mathcal{F}_n)$ must be precisely defined.

Fortunately it is the orthogonal projection if we restrict the large Banach space $L^1(\Omega, \mathcal{F}, P)$ on the dense linear subspace, and Hilbert space,

$$L^2(\Omega, \mathcal{F}, P) \subseteq L^1(\Omega, \mathcal{F}, P).$$

See example [8, Chapter 5].

More precisely we have $E(.|\mathcal{F}_n)$ restricted to $L^2(\Omega, \mathcal{F}, P)$ is the orthogonal projection onto the closed subspace $L^2(\Omega, \mathcal{F}_n, P)$,

$$E(.|\mathcal{F}_n)|_{L^2(\Omega,\mathcal{F},P)} : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{F}_n, P).$$

Throughout this thesis we will denote this restriction of $E(.|\mathcal{F}_n)$ by P_n. So let us restate that we have a projection

$$P_n : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{F}_n, P) \subseteq L^2(\Omega, \mathcal{F}, P).$$

Illustratively we state ($\forall n \in \mathbb{N}$)

$$\boxed{P_n \sim E(.|\mathcal{F}_n).} \tag{1.1}$$

Since this is an **orthogonal projection**, P_n as continuous linear operator acting on the Hilbert space $L^2(\Omega, \mathcal{F}, P)$ satisfies for all $f, g \in L^2(\Omega, \mathcal{F}, P)$ and all scalars $\alpha, \beta \in \mathbb{C}$

$$P_n(\alpha f + \beta g) = \alpha P_n(f) + \beta P_n(g) \textbf{ linear},$$

$$\|P_n(f)\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \textbf{ bounded with norm } 1,$$

$$P_n(P_n(f)) = P_n(f) \textbf{ idempotent} \text{ and}$$

$$< P_n(f), g >=< f, P_n(g) > \textbf{ selfadjoint}.$$

For a projection acting on a Hilbert space one always assumes these four properties. Here $< \cdot, \cdot >$ denotes the inner product in $L^2(\Omega, \mathcal{F}, P)$,

$$< f, g >:= \int_\Omega f(\omega)\overline{g(\omega)}d\omega \quad \text{for all } f, g \in L^2(\Omega, \mathcal{F}, P).$$

Note that we have

$$E(E(.|\mathcal{F}_k)|\mathcal{F}_n) = E(.|\mathcal{F}_k) \text{ for all } k \leq n \in \mathbb{N}.$$

If we restate this fact with the P_n-operators then we have

$$P_k(P_n(f)) = P_k(f) \text{ for all } k \leq n \in \mathbb{N}.$$

Let us return to the Doob-decomposition. It is unique up to the constant processes, i.e. if $X = M + A$ is a particular Doob-decompostion then

$$X = (M + c) + (A - c)$$

gives all possible Doob-decompositions, where c is a scalar.

It is natural to ask wether the Doob-decomposition can be generalized to continuous time processes of the form $X : \mathbb{R}_+ \to L^1(\Omega, \mathcal{F}, P)$. The first who gave an answer to this question was Meyer [13].

Firstly let us give the definition of a submartingale. A **submartingale** is an adapted process

$$X : \mathbb{R}_+ \to L^1(\Omega, \mathcal{F}, P)$$

which satisfies

$$E(X_t|\mathcal{F}_s) \geq X_s \quad \text{for all } s \leq t.$$

We note here that if we apply the discrete time Doob-decomposition on a submartingale, then the resulting predictable process A is increasing, that is

$$A_k \leq A_n \quad \text{a.e. for all } k \leq n \in \mathbb{N}.$$

This follows easily from an explicit formula for the Doob-decomposition which we shall state below.

The **Doob-Meyer-decomposition**, see [13, Theorem 2], states the following: If

$$X : \mathbb{R}_+ \to L^1(\Omega, \mathcal{F}, P)$$

is a pathwise right continuous submartingale of class (D) ([13, Definition 1]), then there exists a decomposition

$$X = M + A$$

where M is martingale and A is predictable and increasing, i.e.

$$A_s \leq A_t \quad \text{a.e. for all } s \leq t \in \mathbb{R}_+.$$

Actually here the predictability became clear after Doleans work [3].

We remark that **positive submartingales fall into class (D)**, see [13, Prop. 1].

In our work we replace the space $L^2(\Omega, \mathcal{F}, P)$ by an arbitrary Hilbert space $\mathcal{H}$. Further we replace the common time scale $\mathbb{R}_+$ by an arbitrary totally ordered set $\mathcal{R}$. A given family of increasing projections $(P_t)_{t\in\mathcal{R}}$ will then play the part of a filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$, i.e. we assume

$$P_sP_t = P_s \text{ (one also writes } P_s \le P_t) \text{ for all } s \le t \in \mathcal{R}.$$

We note that in our situation P_s and P_t commute since they are comparable,

$$P_sP_t = P_s = P_s^* = (P_sP_t)^* = P_t^*P_s^* = P_tP_s.$$

If we have given a filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$, by recalling (1.1), we will define P_t as the operator $E(.|\mathcal{F}_t)$ acting on $L^2(\Omega, \mathcal{F}, P)$.

In the following table we sum up the relations:

In this thesis	**Concrete example**
Hilbert space $\mathcal{H}$	Hilbert space $L^2(\Omega, \mathcal{F}, P)$
Totally ordered set $\mathcal{R}$	$\mathbb{N}, \mathbb{Q}, \mathbb{R}_+$
Projections $(P_t)_{t\in\mathcal{R}} \uparrow$ in $B(\mathcal{H})$	Filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$
$P_t : \mathcal{H} \to \mathcal{H}$	$E(.\|\mathcal{F}_t) : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{F}, P)$

Table 1.1: Generalizations in this thesis.

Here $\mathcal{H}$ can also stand for $L^2(\Omega, \mathcal{F}, P, h)$ (vector-valued integration in the sense of Bochner) where h is a Hilbert space and thus our proofs include infinite dimensional valued stochastic processes.

We shall give two proofs and one application of the Doob-Meyer-decomposition:

1. **Doob-Meyer-decomposition variant 1.** The first proof is very short but already full of power (Theorem 2.2.2). We call it the discrete Doob-Meyer-decomposition since the time scale is endowed with the dicrete topology. (So discrete does not mean countable here.)

2. **Doob-Meyer-decomposition variant 2.** The second proof is considerably longer, see Theorem 2.3.4. This variant is a kind of measurable and integrable version of the first discrete time scaled Doob-Meyer-decomposition.
   Here we assume that the time scale $\mathcal{R}$ is endowed with some measure μ. (E.g. the Lebesgue measure on the time scale $[0, 1]$.)

3. **Application: Doleans-measure and stochastic integration.** As an application we show that a stochastic integral could be defined for more general time scales than $\mathbb{R}_+$ by reproving a theorem of Doleans, who considered some measure on the σ-algebra of predictable sets (Theorem 2.4.1 ff.).

All given proofs are completely self-contained. De facto only very little knowledge on stochastic processes are needed. Points 1 and 3 already completely introduce the Stieltjes-like stochastic integral, even for infinite dimensional (more precisely: arbitrary Hilbert space) valued stochastic processes, over a martingale as integrator.

The reader of the first block of this thesis is assumed to be acquainted with the following techniques:

- basic definitions of stochastic processes and martingales,
  discrete Doob-decomposition

- basic facts about Hilbert spaces;
  weak topology of a Hilbert space, weak-compactness of the unit ball

- ultrafilter

- elementary measure theory
  vector integration; otherwise one restricts to $\mathbb{R}$ or $\mathbb{C}$ valued processes

We shall now discuss each of the three enumerated points listed above.

1.1.1 Doob-Meyer-decomposition variant 1

At first consider a stochastic process $X : \mathbb{N} \to L^2(\Omega, \mathcal{F}, P)$. Then we consider the **Doob-composition** of X which can be computed explicitly as follows:

$$A_n := \sum_{k=1}^{n} E(X_k|\mathcal{F}_{k-1}) - X_{k-1} \text{ and}$$

$$M_n := X_n - A_n.$$

So we obtain $X = M + A$, M is a martingale and A is predictable.

Note that in our general setting we consider $X : \mathbb{N} \to \mathcal{H}$ and compute analogously

$$A_n := \sum_{k=1}^{n} P_{k-1}X_k - X_{k-1} \text{ and}$$
$$M_n := X_n - A_n.$$

Recall here that P_t is the $E(.|\mathcal{F}_t)$-operator ((1.1), table 1.1), i.e. we have

$$P_t f = E(f|\mathcal{F}_t) \text{ for all } f \in \mathcal{H} = L^2(\Omega, \mathcal{F}, P).$$

X is adapted means that $E(X_n|\mathcal{F}_n) = X_n$ $(n \geq 1)$. Analogously for the general setting we call a process $X : \mathbb{N} \to \mathcal{H}$ **adapted** if

$$P_n X_n = X_n \text{ for all } n \geq 1.$$

A is predictable means that $E(A_n|\mathcal{F}_{n-1}) = A_n$ $(n \geq 1)$. Analogously for the general setting we call a process
$A : \mathbb{N} \to \mathcal{H}$ **predictable** if

$$P_{n-1}A_n = A_n \text{ for all } n \geq 1. \tag{1.2}$$

M is a martingale means that $E(M_n|\mathcal{F}_k) = M_k$ $(n \geq k)$. Analogously for the general setting we say a process $M : \mathbb{N} \to \mathcal{H}$ is a **martingale** if

$$P_k M_n = M_k \text{ for all } n \geq k.$$

The definitions of adapted processes and martingale processes can be directly adapted to arbitrary time scales $\mathcal{R}$, i.e. where $\mathcal{R}$ is an arbitrary totally ordered set.

The definition of a predictable process can not be adapted one-to-one since a point of time t in general has no predecessor $s = t - 1$.

Therefore we say a process $A : \mathcal{R} \to \mathcal{H}$ is predictable, if for every point of time $t \in \mathcal{R}$ the value A_t is in the smallest Hilbert space $\mathcal{O}_t$ which contains all Hilbert spaces $\mathcal{H}_s$, where $s < t$. Here we let $\mathcal{H}_s$ be the Hilbert space associated to the projection P_s, i.e.

$$\mathcal{H}_s := \mathrm{Im}(P_s) \subseteq \mathcal{H}.$$

And we clearly obtain the explicit formula for this smallest Hilbert space

$$\mathcal{O}_t = \overline{\bigcup_{s \in \mathcal{R}, s < t} \mathcal{H}_s},$$

where the line here denotes the norm-closure. Clearly we have

$$\mathcal{H}_s \subseteq \mathcal{O}_t \subseteq \mathcal{H}_t \quad \text{for all } s < t \in \mathcal{R}.$$

If we associate with the Hilbert space $\mathcal{O}_t$ the projection $L_t : \mathcal{H} \to \mathcal{H}$ (L for *left*), which let be the projection onto $\mathcal{O}_t$, then we obtain the definition: A is **adapted** if

$$L_t A_t = A_t \text{ for all } t \in \mathcal{R}. \tag{1.3}$$

As an example let us compute L_n for the time scale $\mathbb{N}$.

$$\mathcal{O}_n = \overline{\cup_{k<n} \mathrm{Im}(P_n)} = \mathrm{Im}(P_{n-1}).$$

Thus L_n, per def. the projection onto $\mathcal{O}_n = \mathrm{Im}(P_{n-1})$, is P_{n-1}, since P_{n-1} projects onto $\mathrm{Im}(P_{n-1})$ and there exists only *one* orthogonal (since selfadjoint as we always assume) projection for each Hilbert subspace. So $L_n = P_{n-1}$ and the definition (1.3) is consistent with the definition (1.2).

In Lemma 2.2.1 we compute the norms of A_n and M_n for every point of time n of the Doob-decomposition. In order to go not to much into details in this introduction, let us restrict to the case that the process X is a positive submartingale, i.e. $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$ and for all $t \in \mathcal{R}$ we assume

$$X_t \geq 0 \quad \text{a.e. and}$$

$$E(X_t|\mathcal{F}_s) \geq X_s \quad \text{a.e. for all } s \leq t.$$

Then it turns out (lemma 2.2.1) that for the martingale part of the Doob-decomposition we have

$$\|M_n\|_{\mathcal{H}} \leq \|X_n\|_{\mathcal{H}} \quad \text{ for all } n \geq 1.$$

That is fundamental since it tells us that **the processes A and M of the Doob-decomposition do not explode in L^2-norm**.

Indeed for the predictable part we have

$$\|A_n\|_{\mathcal{H}} = \|X_n - M_n\|_{\mathcal{H}} \leq 2\|X_n\|_{\mathcal{H}} \text{ for all } n \geq 1.$$

It is fundamental here that the estimate on $\|M_n\|$ and $\|A_n\|$ do not depend on the whole past process $(X_0, X_1, X_2, ..., X_n)$ but only on the "current" value $\|X_n\|$.

Beside, let us note that the estimate

$$\|A_n\|_{L^1(\Omega,\mathcal{F},P)} \leq \|X_n\|_{L^1(\Omega,\mathcal{F},P)}$$

is trivially for the L^1-norm. But the L^1-norm has the failure that $L^1 \neq (L^\infty)^*$ in general, otherwise we could apply the concept that follows to L^1-processes.

Now in full generality on the time scale $\mathcal{R}$, let be given a submartingale for a totally ordered set $\mathcal{R}$,

$$X : \mathcal{R} \to L^2(\Omega, \mathcal{F}, P).$$

We are given a filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$, i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t \in \mathcal{R}$. We choose an arbitrary finite set of points of time

$$u = \{t_0 < t_1 < t_2 < ... < t_N\} \subseteq \mathcal{R}$$

and consider the finite process $(X_{t_0}, X_{t_1}, X_{t_2}, ..., X_{t_N})$. It is a submartingale since this is true for the whole process X. We apply the Doob-decomposition on this finite process and obtain a predictable process $(A_{t_0}, A_{t_1}, A_{t_2}, ..., A_{t_N})$, i.e.

$$P_{t_{k-1}} A_{t_k} = A_{t_k} \text{ for all } 1 \leq k \leq N,$$

and a martingale $(M_{t_0}, M_{t_1}, M_{t_2}, ..., M_{t_N})$, i.e.

$$P_{t_k} M_{t_n} = M_{t_k} \text{ for all } 1 \leq k \leq n \leq N,$$

such that

$$X_{t_k} = A_{t_k} + M_{t_k} \text{ for all } 1 \leq k \leq N.$$

Beside, since X is a submartingale, the finite process A is increasing, i.e.

$$A_{t_0} \leq A_{t_1} \leq ... \leq A_{t_N} \text{ a.e.}$$

This follows easily from the formula of the Doob-decomposition.

Recall the bijection between the Hilbert space $\mathcal{H}$ and its dual space $\mathcal{H}^*$, i.e. the set of all linear functionals which are bounded with respect to the Hilbert space norm,

$$\mathcal{H} \leftrightarrow \mathcal{H}^*.$$

The pointwise convergence of the function space $\mathcal{H}^*$ induces the weak topology. By a well-known theorem of Alaoglu the unit ball

$$\{\, \xi^* \mid \|\xi^*\|_{\mathcal{H}^*} \leq 1 \,\}$$

is compact under the weak* topology, which coincides with the weak topology in the Hilbert space if we identify $\mathcal{H}$ with $\mathcal{H}^*$ via the stated antilinear bijection.

Summing up these considerations we have the following fact: Under the **weak convergence** on $\mathcal{H}$, i.e.

$$\text{a net converges weakly } (\xi_i) \to \xi \quad :\Leftrightarrow$$

$$\forall \eta \in \mathcal{H} : \lim_i < \xi_i, \eta > = < \xi, \eta >,$$

the unit ball $\{\, \xi \in \mathcal{H} \mid \|\xi\|_{\mathcal{H}} \leq 1 \,\}$ is compact (with respect to the weak topology).

By a suited choice of an ultrafilter on the sets of end-pieces of the set of the finite subsets of $\mathcal{R}$ we then use the weak topology on $\mathcal{H}$ and our bounds

$$\|M_n\| \leq \|X_n\| \quad \text{and} \quad \|A_n\| \leq 2\|X_n\|$$

for compactness to get accumulation points M_t and A_t in $\mathcal{H}$ for each point of time $t \in \mathcal{R}$. More detailed for each sample of finite many points of time

$$u = (t_0 < t_1 < ... < t_k = t < t_{k+1} < ... < t_N)$$

we get samples $M_t(u)$ and $A_t(u)$ due to the Doob-decomposition on

$$(X_{t_0}, X_{t_1}, ..., X_{t_N}).$$

The set of all these samples $M_t(u), A_t(u)$ form the accumulation points M_t and A_t.

It turns out that this vectors M_t and A_t serves the desired Doob-Meyer-decompostion

$$X = M + A.$$

Let us emphasize that we decompose the process X for each point of time t on its own, i.e.

$$X_t = M_t + A_t.$$

But since we use *one common* ultrafilter for all point of times, this ultrafilter forms the cement such that (M_t) is a martingale and (A_t) is a increasing and predictable process.

Actually we do not decompose only submartingales but the largest set of Hilbert space valued functions $f : \mathcal{R} \to \mathcal{H}$ for which we can apply our concept of proof. By an (though somewhat primitive) example we demonstrate that there are more such functions f which could be decomposed but which do not fall into the set of function we decompose by theorem. Also we demonstrate that the predictable part A need not have bounded variation.

We stress that **none of our functions $f : \mathcal{R} \to \mathcal{H}$ are assumed to be right continuous.** This is a strong improvement to the variants of the Doob-Meyer-decomposition we know, since there right continuity is always assumed.

Nevertheless using "Doob's regularization theorem", it is not hard to show the following. If one has a Doob-Meyer-decomposition $X = M + A$ of a pathwise right-continuous submartingale $X : \mathbb{R}_+ \to L^1(\mathbb{P})$ (and the filtration let be r.c. too) into a martingale M and an increasing process A (i.e. $A_s \leq A_t$

a.e. for all $s \leq t$), then one finds pathwise right-continuous versions M' and A' of M and A such that $X = M' + A'$ where M' is (clearly) a martingale and A' is pathwise increasing.

We pointed out this fact since the increase of a process is never meant pathwise in this work; but that is not so "bad" as we have just seen with "Doob's regularization".

Finally let us state the discussed result, Theorem 2.2.2

Theorem 1.1.1 *Let $\mathcal{R}$ be a totally ordered set. Let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ an increasing family of projections in $B(\mathcal{H})$. Let $X : \mathcal{R} \to \mathcal{H}$ an adapted function. For all $t \in \mathcal{R}$ let exist a constant $C_t \in \mathcal{R}$ such that for all $n \geq 1$ and all $t_0 < ... < t_n = t$ in $\mathcal{R}$ we have*

$$\sum_{k=1}^{n} \|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2 \geq C_t. \tag{1.4}$$

Then we find adapted functions $M, A : \mathcal{R} \to \mathcal{H}$ such that $X = M + A$ and M is a martingale and A is predictable.

If $\mathcal{H} = L^2(\Omega)$ and X is a positive submartingale then A can be chosen increasing (i.e. $A_s \leq A_t$ for all $s \leq t$ in $\mathcal{R}$).

1.1.2 Doob-Meyer-decomposition variant 2

Roughly speaking the Doob-Meyer-decomposition variant 2 (Theorem 2.3.4) is the following: Consider a measure μ on $\mathbb{R}_+$ such that

$$\mu(\mathbb{R}_+) < \infty.$$

In example one may take

$$\mu((a, b]) = \frac{b}{1+b} - \frac{a}{1+a}.$$

Consider a positive submartingale

$$X : \mathbb{R}_+ \to L^2(\Omega, \mathcal{F}, P)$$

which is right continuous in the following sense:

$$\lim_{t\downarrow s}\int X(t)Y\,dP=\int X(s)Y\,dP$$

for all $s\in\mathcal{R}$ and all $Y\in L^2(\Omega,\mathcal{F},P)$.

Furthermore assume that X regarded as a function

$$X:\mathbb{R}_+\times\Omega\to\mathbb{R}$$

is measurable with respect to the $\mathcal{B}\times\mathcal{F}$ σ-algebra on $\mathbb{R}_+\times\Omega$ and that X is quadratically integrable, i.e.

$$X\in\mathcal{H}:=L^2(\mathbb{R}_+\times\Omega,\mathcal{B}\otimes\mathcal{F},\mu\otimes P).$$

Then the Doob-Meyer-decomposition variant 2 gives a decomposition

$$X=M+A\quad\mu\otimes P-\text{a.e.}$$

such that M and A are also in the space $\mathcal{H}$ and also fulfill the right continuity properties we assumed on X. Actually M is even right continuous in a stronger sense. Clearly M is here a martingale and A is predictable and increasing.

In other words and terms, using vector integration (in the sense of Bochner, e.g. see [11, Chapter VI, General Integral]), our theorem states that any positive submartingale

$$X\in\mathcal{H}:=L^2(\mathbb{R}^+,\mathcal{B},\mu,L^2(\Omega,\mathcal{F},P)),$$

which is right continuous w.r.t. the weak topology of $L^2(\Omega,\mathcal{F},P)$, has a decomposition

$$X=M+A\quad\mu-\text{a.e.},$$

where also M and A are in the space $\mathcal{H}$ and M is a martingale and A is increasing and predictable (in the common sense also as in the sense of the previous chapter since both definitions of predictability coincides here, Lemma 2.3.2).

Actually, as in the previous chapter, we replace the Hilbert space $L^2(\Omega, \mathcal{F}, P)$ by an arbitrary Hilbert space $\mathcal{H}$ and a filtration $(\mathcal{F}_t)_t$ by an increasing family of projections $(P_t)_t$ in $B(\mathcal{H})$. Additionally the family of projections $(P_t)_t$ is here assumed to be right continuous in a certain sense. It is the counterpart to the right continuity of a filtration.

Once more we replace the common time scale $\mathbb{R}_+$ by a totally ordered set $\mathcal{R}$. But now in contrary to the previous chapter we assume that the time scale $\mathcal{R}$ satisfies the following two properties:

- $\mathcal{R}$ is (topologically) separable.
- $\mathcal{R}$ has at most countably many successor elements.

Here we think of $\mathcal{R}$ endowed with the order topology, i.e. the topology induced by the (per def.) open sets

$$\{\, t \in \mathcal{R} \mid s < t \,\} \text{ and } \{\, t \in \mathcal{R} \mid t < s \,\}.$$

Recall that $t \in \mathcal{R}$ is a **successor** of an element $s \in \mathcal{R}$, if

$$(s,t) \cap \mathcal{R} = \emptyset$$

where (s,t) is the open interval

$$(s,t) = \{\, x \in \mathcal{R} \mid s < x < t \,\}.$$

Any subset of $\mathbb{R}$ fulfills the properties we assume on $\mathcal{R}$.

We can decompose more processes than positive submartingales, in example all processes which fulfill

$$\|P_s X_t\|_{\mathcal{H}} \geq \|X_s\|_{\mathcal{H}} \quad \text{for all } s < t \in \mathcal{R}.$$

This property obviously looks like the generalized counterpart of a submartingale. These processes can also be decomposed with the discrete Doob-Meyer-decomposition variant 1.

But a somewhat primitive example shows once again that our theorem does not decompose all possible candidates.

We briefly give an idea of the proof of the Doob-Meyer-decomposition variant 2:

Once again we use the weak topology of a Hilbert space, the compactness of the unit ball and the fundamental estimate

$$\|M_n\| \leq \|X_n\|$$

of the Doob-decomposition. But now in contrast to the previous chapter we do not compute an accumulation point M_t for each point of time t separately, but compute *one* accumulation point M in the Hilbert space

$$L^2(\mathcal{R} \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \otimes P).$$

If for example we have given a *discrete* measure μ, lets say the measure

$$\mu(t_n) := 2^{-n}$$

for an enumeration

$$\{t_1, t_2, t_3, ...\} = \mathbb{Q}$$

of the time scale $\mathbb{Q}$, then using the fundamental estimate $\|M_n\| \leq \|X_n\|$ it is relatively easy to prove the theorem. Consider the Doob-decomposition of the finite process

$$(X_{t_1}, X_{t_2}, ..., X_{t_N})$$

and obtain a martingale

$$(M_{t_1}, M_{t_2}, ..., M_{t_N}).$$

This martingale can be easily extended to a martingale

$$\tilde{M}_N : \mathbb{Q} \to \mathbb{R}.$$

Then we get an accumulation point M for the sequence of martingales

$$(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, ...)$$

in the space

$$\mathcal{H} := L^2(\mathbb{Q} \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \otimes P).$$

This vector M must also be a martingale since the space of martingales is a norm closed subspace in $\mathcal{H}$ and hence weakly closed and the accumulation point M is within the weakly closed subspace of martingales.

In this manner one gets a decomposition of X.

Now if we have given an arbitrary finite measure μ which is not necessarily discrete, then in principle the idea of the proof is like the outlined proof for a discrete measure μ, but in this case the proof is in contrast considerable more technically. Instead of a sum of delta functions

$$\sum_{i=1}^{\infty} \alpha_{t_i} \delta_{t_i}$$

one has to use step functions of the form

$$\sum_{i=1}^{N} \alpha_{t_i} 1_{(t_{i-1},t_i]}.$$

But this immediately makes a problem: in general a martingale can not be a step function. So this step function must be smoothed with the $E(.|\mathcal{F}_t)$-operator to get a martingale. But is that function measurable? What about its norm. Does the smoothed version now "explode" in norm? I.e. can it happen that

$$\|M_n\|_{L^2(\mathbb{Q}\times\Omega,\mathcal{B}\otimes\mathcal{F},\mu\otimes P)} \to \infty$$

the finer we make the Doob-decomposition? And so on.

Additionally the measure μ can be quite unusual because it can be a mix of point masses and smooth measure. For example consider the measure

$$\mu = \lambda + \nu$$

where λ is the Lebesgue measure on the interval $[0, 1]$ and ν is a discrete measure on $\mathbb{Q} \cap [0, 1]$. A measure, which our theorem also takes into account since our only restriction on μ is finiteness,

$$\mu(\mathcal{R}) < \infty.$$

We state here the result of this section, Theorem 2.3.4.

Theorem 1.1.2 *Let $\mathcal{R}$ be a totally ordered set with finite measure μ and assume (I). Let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ an increasing, right continuous family of projections in $B(\mathcal{H})$. Let $X \in \mathcal{L}^2(\mathcal{R}, \mu, \mathcal{H})$ be adapted and right continuous w.r.t. the weak topology in $\mathcal{H}$ and let exist a constant $C \in \mathbb{R}$ such that $\forall n \geq 1$ $\forall t_0 < ... < t_n \in \mathcal{R}$*

$$\sum_{k=1}^{n} \|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2 \geq C.$$

Then we find $M, A \in \mathcal{L}^2(\mathcal{R}, \mu, \mathcal{H})$ such that $X = M + A$ μ-a.e., such that M is a right continuous martingale μ-a.e. and such that A is predictable μ-a.e.

If $\mathcal{H} = L^2(\Omega)$ and X is a positive submartingale then A can be chosen increasing, i.e. $A_s \leq A_t$ for all $s \leq t$ in $\mathcal{R}\backslash Z$ for some μ-null set Z.

1.1.3 Doleans measure and stochastic integration

For a totally ordered set $\mathcal{R}$ and probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$ consider the **predictable σ-algebra** $\mathcal{P}$ on

$$\mathcal{R} \times \Omega$$

which is defined to be induced by the predictable rectangles

$$\boxed{(s,t] \times C \quad \text{where } s < t \in \mathcal{R} \text{ and } C \in \mathcal{F}_s.}$$

Consider the subalgebra

$$\mathcal{P}_0 \subseteq \mathcal{P}$$

which consists of those elements of $\mathcal{P}$, which are finite unions of disjoint predictable rectangles,

$$\bigcup_{k=1}^{N} (s_k, t_k] \times C_k.$$

Now let us given a submartingale

$$X : \mathcal{R} \to L^1(\Omega, \mathcal{F}, P).$$

Then elementary calculations show that the (additively defined) function

$$\mu : \mathcal{P}_0 \to \mathbb{R}_+$$

$$\mu((s,t] \times C) = \int (X_t - X_s) 1_C dP$$

is well defined. So for example if

$$(s_3, t_3] \times C_3 = \Big((s_1, t_1] \times C_1\Big) \cup \Big((s_2, t_2] \times C_2\Big)$$

is a disjoint partition of a predictable rectangle into two smaller ones, then we have

$$\begin{aligned} \mu((s_3, t_3] \times C_3) &= \mu((s_1, t_1] \times C_1) \\ &\quad + \mu((s_2, t_2] \times C_2), \end{aligned}$$

that is

$$\begin{aligned} \int (X_{t_3} - X_{s_3}) 1_{C_3} dP &= \int (X_{t_1} - X_{s_1}) 1_{C_1} dP \\ &\quad + \int (X_{t_2} - X_{s_2}) 1_{C_2} dP. \end{aligned}$$

The question here is wether the function μ is even σ-additive (on $\mathcal{P}_0$) and thus could be extended to the predictable σ-algebra $\mathcal{P}$ via measure extension.

If X is right continuous, this is indeed the case. The function μ was discovered and the σ-additivity was proved by Mrs. Doleans [4].

We give here a proof of the Doleans measure in Theorem 2.4.1 for positive L^2-submartingales and for a more general time scale than $\mathbb{R}_+$.

In more details let us given a totally ordered set $\mathcal{R}$. Then we assume that $\mathcal{R}$ is **sequentially right complete**. That means that any *countable* subset

$$C \subseteq \mathcal{R}$$

which is bounded from below, i.e.

$$\exists M \in \mathcal{R} \text{ such that } M \leq c \text{ for all } c \in C,$$

has an infimum (in $\mathcal{R}$), i.e.

$$\exists s \in \mathcal{R} \text{ such that } s = \inf(C).$$

Furthermore we say a function

$$f : \mathcal{R} \to Y$$

into a topological space Y is **sequentially right continuous**, if f is right continuous in all points which are accumulation points of a sequence from the right, i.e. if f is right continuous in all points of the set

$$\mathcal{S} = \{\, s \in \mathcal{R} \mid \exists \text{ countable set } C \subseteq \mathcal{R} \text{ such that } s = \inf(C) \,\}.$$

In our theorem we prove then the following: If

$$X : \mathcal{R} \to L^2(\Omega, \mathcal{F}, P)$$

is a positive submartingale on a sequentially right complete totally ordered set $\mathcal{R}$, such that the function

$$\mathcal{R} \to \mathbb{R}_+ : t \mapsto \mathbb{E}(X_t)$$

is sequentially right continuous, then the function

$$\mu((s,t] \times C) := \mathbb{E}((X_t - X_s)1_C)$$

defined on the predictable rectangles can be extended to a measure μ on the predictable σ-algebra $\mathcal{P}$.

We state here our result, Theorem 2.4.1.

Theorem 1.1.3 *Let $\mathcal{R}$ be a totally ordered, sequentially right complete set. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$. Let $X : \mathcal{R} \to L^2(\Omega)$ a positive submartingale, such that $t \to E(X_t)$ is sequentially right continuous. Then*

$$\mu(C \times (s,t]) = E(1_C\,(X_t - X_s)),$$

for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$, defines a measure on the predictable σ-algebra $\mathcal{P}$.

As a very useful application of this measure consider a martingale

$$M : \mathcal{R} \to L^4(\Omega, \mathcal{F}, P).$$

Then

$$M^2 : \mathcal{R} \to L^2(\Omega, \mathcal{F}, P)$$

is a positive submartingale. If the filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$ is right continuous, the martingale M and hence the submartingale M^2 is right continuous.

Our theorem is then applicable on M^2 and we obtain a measure

$$\begin{aligned}
\mu_M((s,t] \times C) &= \mathbb{E}((M_t^2 - M_s^2)1_C) \\
&= \|M_t 1_C\|_{L^2}^2 - \|M_s 1_C\|_{L^2}^2 \\
&= \|(M_t - M_s)1_C\|_{L^2}^2.
\end{aligned}$$

on the predictable σ-algebra $\mathcal{P}$. Actually we prove this fact for L^2-valued martingales M, see Lemma 2.4.3. (Beside, we have "discovered" and proved that measure before we heard about the already well-known Doleans-measure.)

The measure μ_M can now usefully applied for the definition of the **stochastic integral**. Consider a process

$$f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$$

which is indeed just the characteristic function of a predictable rectangle, i.e.

$$f(t, \omega) = 1_{(a,b]\times C}(t, \omega),$$

where $a < b$ are two points of time and $C \subseteq \Omega$ is $\mathcal{F}_a$-measurable. Then one defines the stochastic integral for the martingale M as integrator via

$$\int_{\mathbb{R}_+} f(t, \omega) dM_t(\omega) := (M_b(\omega) - M_a(\omega))1_C(\omega).$$

More precisely one defines for a L^2-martingale M

$$\int_{\mathbb{R}_+} .dM : \text{step}(\mathbb{R}_+ \times \Omega, \mathcal{P}_0) \to L^2(\Omega, \mathcal{F}, P)$$

$$\int_{\mathbb{R}_+} 1_{(a,b]\times C} dM := (M_b - M_a)1_C.$$

Here step$(\mathbb{R}_+ \times \Omega, \mathcal{P}_0)$ denotes the linear span of the characteristic functions of the predictable rectangles, where $\mathcal{P}_0$ denotes the set algebra of predictable rectangles.

Now we have that measure μ_M on $\mathcal{P}$ and we can form the space

$$L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mu_M).$$

By a simple calculation one can prove that for a predictable step function

$$f = \sum_{k=1}^{n} \alpha_k 1_{A_k},$$

where A_k are disjoint predictable rectangles and α_k are scalars, one has

$$\begin{aligned}
\Big\| \int f dM \Big\|_{L^2(\Omega,\mathcal{F},P)} &= \mathbb{E}\Big(\Big| \int \sum_{k=1}^{n} \alpha_k 1_{A_k} dM \Big|^2\Big) \\
&= \sum_{k=1}^{n} |\alpha_k|^2 \mathbb{E}\Big(\Big| \int 1_{A_k} dM \Big|^2\Big) \\
&= \sum_{k=1}^{n} |\alpha_k|^2 \mu_M(A_k) \\
&= \Big\| \sum_{k=1}^{n} \alpha_k 1_{A_k} \Big\|_{L^2(\mathbb{R}_+\times\Omega,\mathcal{P},\mu_M)} \\
&= \|f\|_{L^2(\mathbb{R}_+\times\Omega,\mathcal{P},\mu_M)}.
\end{aligned}$$

Here one uses the fundamental orthogonality for disjoint predictable rectangles,

$$A_1 = (s_1, t_1] \times C_1, \quad A_2 = (s_2, t_2] \times C_2,$$

$$\begin{aligned}
\Big\| \int 1_{A_1} + 1_{A_2} dM \Big\|_{L^2} &= \Big\| (M_{t_1} - M_{s_1})1_{C_1} + (M_{t_2} - M_{s_2})1_{C_2} \Big\|_{L^2} \\
&= \Big\| (M_{t_1} - M_{s_1})1_{C_1} \Big\|_{L^2} + \Big\| (M_{t_2} - M_{s_2})1_{C_2} \Big\|_{L^2},
\end{aligned}$$

which is indeed just the additivity of μ_M since

$$\mu_M(1_A) = \|(M_t - M_s)1_A\|_{L^2}.$$

This property of orthogonality we consider as *the* reason why stochastic integration over L^2-martingales (and indeed this is the only possible integrator class beside the processes with bounded variation where one can take the Stieltjes' concept) works.

Now for step functions f on predictable rectangles we have shown

$$\Big\| \int f dM \Big\|_{L^2(\Omega,\mathcal{F},P)} = \|f\|_{L^2(\mathbb{R}_+\times\Omega,\mathcal{P},\mu_M)}$$

whence our stochastic integral for step functions on predictable rectangles is an isometry and thus can be isometrically extended to an isometry

$$\int_{\mathbb{R}_+} .dM : L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mu_M) \to L^2(\Omega, \mathcal{F}, P).$$

Somewhat more generally we define a stochastic integral

$$\int .dM : L^2(\mathcal{R} \times \Omega, \mathcal{P}, \mu_M, \mathcal{H}) \to L^2(\Omega, \mathcal{F}, P),$$

where $\mathcal{H}$ is an arbitrary Hilbert space, M is a $\mathcal{H}$-valued martingale

$$M : \mathcal{R} \to L^2(\Omega, \mathcal{F}, P, \mathcal{H}),$$

$\mathcal{R}$ is a sequentially right complete totally ordered set and the filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$ is right continuous.

One puts

$$\Big(\int 1_{(s,t]\times C}\xi dM\Big)(\omega) :=< \xi, (M_t - M_s)(\omega)1_C(\omega) >_{\mathcal{H}}$$

for $\xi \in \mathcal{H}$.

For example if $\mathcal{H} := \mathbb{R}^n$ with the usual inner product, then here the more dimensional stochastic integral integrates each coordinate and then takes the sum. So, in financial terms lets say, it gives the whole wealth by summing up all n single wealths.

1.2 Annihilation of subspaces in $C(K)$ and the fundamental theorem of asset pricing

This part of the thesis is motivated by the fundamental theorem of asset pricing.

Consider a **finite** probability space Ω, i.e. $\mathrm{card}(\Omega) < \infty$.

Let $X : \mathbb{R}_+ \to L^\infty(\Omega, \mathcal{F}, P)$ be an adapted process, e.g. an asset in financial terms.

Then the question is under which conditions on X we find a probability measure $\hat{P}$ on Ω such that $\hat{P}$ does neither loose nor "invent" information w.r.t the measure P, i.e. $\hat{P}(\{\omega\}) = 0$ if an only if $P(\{\omega\}) = 0$ $(\forall \omega \in \Omega)$ and such that the process X becomes a martingale under the probability measure $\hat{P}$, i.e.

$$\hat{\mathbb{E}}(X_t | F_s) = X_s \quad (\forall t \geq s).$$

This equation can also be expressed in the following way:

$$0 = \hat{\mathbb{E}}(X_t - X_s | \mathcal{F}_s) \quad (\forall t \geq s),$$

or equivalently

$$\forall A \in \mathcal{F}_s : 0 = \int_\Omega 1_A \hat{\mathbb{E}}(X_t - X_s) d\hat{P} \quad (\forall t \geq s),$$

that is

$$\forall A \in \mathcal{F}_s : 0 = \int_\Omega 1_A (X_t - X_s) d\hat{P} \quad (\forall t \geq s). \tag{1.5}$$

So we can view the problem as follows: We have given a set

$$\Gamma = \{\, 1_A(X_t - X_s) \in l^\infty(\Omega) \mid s < t \in \mathbb{R}_+, A \in \mathcal{F}_s \,\}$$

in the $|\Omega|$-dimensional space

$$l^\infty(\Omega) = \{f : \Omega \to \mathbb{R}\}.$$

In accordance with (1.5) we search scalars $\hat{P}(\{\omega\})$ for all $\omega \in \Omega$, such that for all $f \in \Gamma$ we have

$$\sum_{\omega\in\Omega} f(\omega)\hat{P}(\{\omega\}) = 0,$$

such that

$$\sum_{\omega\in\Omega} \hat{P}(\{\omega\}) = 1,$$

and such that $\hat{P}(\{\omega\}) = 0$ if and only if $P(\{\omega\}) = 0$.

This measure $\hat{P}$ produces a linear functional, the integral operator

$$\phi : l^\infty(\Omega) \to \mathbb{R}$$

$$f \mapsto \sum_{\omega\in\Omega} f(\omega)\hat{P}(\{\omega\}) = \sum_{\omega\in\Omega} f(\omega)\phi(1_{\{\omega\}}).$$

With the relation $\hat{P}(\{\omega\}) = \phi(1_{\{\omega\}})$ we can reformulate our problem as follows.

We search a positive linear functional

$$\phi : l^\infty(\Omega) \to \mathbb{R}$$

such that for all $f \in \Gamma$ we have

$$0 = \phi(f),$$

such that $\phi(1_\Omega) = 1$ and $\phi(1_{\{\omega\}}) = 0$ if and only if $P(\omega) = 0$, for all $\omega \in \Omega$. Indeed we can set $\hat{P}(\{\omega\}) := \phi(1_{\{\omega\}})$ and write

$$\phi(f) = \sum_{\omega\in\Omega} \phi(f(\omega)1_{\{\omega\}}) = \int_\Omega f d\hat{P}.$$

So ϕ defines the probability measure $\hat{P}$.

If we have found the functional ϕ, i.e. $\phi(\Gamma) = 0$, then clearly even $\phi(\text{span}(\Gamma)) = 0$, and vice versa. So it does not matter wether we consider Γ or $\text{span}(\Gamma)$.

Thus let us consider

$$\mathcal{H} := \mathrm{span}(\Gamma)$$

instead of Γ.

Assume ϕ is the desired functional as described. Since $\phi(1_A) = 0$ if $P(1_A) = 0$, ϕ defines a linear positive functional on

$$L^\infty(\Omega, \mathcal{F}, P) = l^\infty(\Omega)/\{\, f \in l^\infty(\Omega) \mid f \text{ is } 0 \ P\text{-a.e.}\,\}.$$

So we obtain a corresponding linear functional

$$\tilde{\phi} : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}.$$

Let us now identify vectors in $l^\infty(\Omega)$ with the corresponding equivalence classes in $L^\infty(\Omega, \mathcal{F}, P)$. Then we have $\tilde{\phi}(\mathcal{H}) = 0$, since $\phi(\mathcal{H}) = 0$.

Since $\phi(1_A) > 0$ if $P(A) > 0$, we get for all $f > 0$ P-a.e.

$$\tilde{\phi}(f) > 0.$$

We sum up: If we have the desired measure $\hat{P}$, then we can define a linear functional

$$\boxed{\begin{gathered} \tilde{\phi} : L^\infty(P) \to \mathbb{R}, \\ \tilde{\phi}(\mathcal{H}) = 0, \\ \tilde{\phi}(1_\Omega) = 1, \\ \tilde{\phi}(f) > 0 \quad \forall f \in L^\infty(P), f > 0. \end{gathered}}$$

On the other hand, given such a functional $\tilde{\phi}$, we just have to define $\hat{P}(A) := \tilde{\phi}(1_A)$ for $A \in \mathcal{F}$ and obtain

$$\int f d\hat{P} = \hat{\phi}(f),$$

and in consequence have the desired positive measure $\hat{P}$.

Consider a functional $\tilde{\phi}$ with the properties listed in the box above. This is a linear functional $\tilde{\phi}$ on $L^\infty(\Omega, \mathcal{F}, P)$ which **annihilates** the subspace $\mathcal{H} \subseteq L^\infty(P)$ and which gives 1 for the one element 1_Ω of the algebra $L^\infty(P)$.

Further this functional is **positive** and even **faithful**, since it does not loose information, i.e. $f > 0$ implies $\tilde{\phi}(f) > 0$.

For an algebra A with unit 1_A a positive linear functional $\tilde{\phi} : A \to \mathbb{R}$ which is scaled such that $\tilde{(}1_A) = 1$ is usually called a **state**.

For a finite probability space $(\Omega, \mathcal{F}, P)$, i.e. $|\Omega| < \infty$, it is not difficult to show the following

Theorem 1.2.1 *If $\mathcal{H} \subseteq L^\infty(\Omega, \mathcal{F}, P)$ is a linear subspace and Ω is finite, then there exists a P-equivalent measure $\hat{P}$ such that $\int \mathcal{H} d\hat{P} = 0$, if and only if*

$$\mathcal{H} \cap L^\infty(P)_+ = \{0\}.$$

So $\mathcal{H}$ must not contain a positive vector except 0.

The case where Ω is arbitrary (not necessarily finite) can also be described. But things are more difficult:

Theorem 1.2.2 (Kreps,Yan) *If $\mathcal{H} \subseteq L^\infty(\Omega, \mathcal{F}, P)$ is a linear subspace, then there exists a P-equivalent measure $\hat{P}$ such that $\int \mathcal{H} d\hat{P} = 0$, if and only if*

$$\overline{\{\, h - f \in L^\infty(P) \mid h \in \mathcal{H}, f \geq 0 \,\}}^{\sigma(L^\infty(P), L^1(P))} \cap L^\infty(P)_+ = \{0\}.$$

So the $\sigma(L^\infty(P), L^1(P))$-closure of the set $\mathcal{H} - L^\infty(P)_+$ must not contain a positive vector except 0.

Our main result is the following middle thing between the last two theorems:

Theorem. If $\mathcal{H} \subseteq L^\infty(\Omega, \mathcal{F}, P)$ is a linear subspace and

$$0 < c \leq 1 \leq C < \infty$$

are real constants, then there exists a P-equivalent measure $\hat{P}$ such that $\int \mathcal{H} d\hat{P} = 0$ and

$$c \leq d\hat{P}/dP \leq C,$$

if and only if

$$\forall x \in \mathcal{H} : \int x dP \leq \inf \Big\{ \frac{C-1}{C}\alpha + \frac{1-c}{C}|\beta| \;\Big|\; \alpha, \beta \in \mathbb{R} \text{ with}$$

$$\int x\, dP \leq \alpha \leq \sup(x) \text{ and}$$

$$\beta \leq 0, \int (x-\alpha) \vee \beta dP = 0 \Big\}.$$

So the vectors of $\mathcal{H}$ must fulfill some "non-positivity" property (which is not so obvious from this inequality).

1.2.1 Problems one has to face

Recalling the considerations above, we may reduce our original question, when do we find an equivalent probability measure Q under which X becomes a martingale, to the following problem:

We have given the space $L^\infty(\Omega, \mathcal{F}, P)$ which we denote simply by L^∞ and a subspace $\mathcal{H} \subseteq L^\infty$.

Then we search a P-equivalent measure Q such that $\int \mathcal{H} dQ = 0$.
(Remember that two measures $P, \hat{P}$ defined on Ω are called **equivalent** if for every measurable $A \subseteq \Omega$ we have $P(A) = 0 \Leftrightarrow \hat{P}(A) = 0$.)

Contrary to the case where the probability space is finite it is in general not enough to just give a positive linear functional

$$\phi : L^\infty \to \mathbb{R}$$

such that $\phi(\mathcal{H}) = 0$ and

$$\phi(f) > 0 \text{ if } f > 0,$$

i.e. ϕ is **positive + faithful**.

In order that ϕ really defines a probability measure it is necessary that it is σ-continuous (such that the measure becomes σ-additive), i.e. if $(f_n)_{n\geq 1}$ is sequence of positive functions in L^∞, decreasing pointwise to 0, i.e. $f_n \downarrow 0$, then we should have

$$\phi(f_n) \downarrow 0.$$

Equivalently we can say that ϕ must be represented by an element of $L^1(\Omega, \mathcal{F}, P)$. Indeed, we know that

$$L^1(\Omega, \mathcal{F}, P) \subseteq L^\infty(\Omega, \mathcal{F}, P)^*.$$

If we have given some $g \in L^1$, then we can interpret g as a continuous linear functional on L^∞,

$$"g" : L^\infty \to \mathbb{R} : f \mapsto \int fgdP.$$

Here we may interpret g as a density function dQ/dP, so this functional g defines a measure on $(\Omega, \mathcal{F})$,

$$Q(A) := \int 1_A gdP \quad (A \in \mathcal{F}),$$

and the functional g can be written as

$$f \mapsto \int fgdP = \int f\frac{dQ}{dP}dP = \int fdQ.$$

Indeed we have the following fact: Each continuous functional $\rho : L^\infty \to \mathbb{R}$ can be uniquely written as a sum

$$\rho = \rho_n + \rho_s$$

where both ρ_n and ρ_s are also continuous linear functionals $\rho_n : L^\infty \to \mathbb{R}$ and $\rho_s : L^\infty \to \mathbb{R}$.

The functional ρ_n is in fact an element of $L^1 \subseteq L^{\infty *}$. ρ_n is called the **normal** part of ρ.

ρ_s is called the **singular** part of ρ.

In general the elements of L^1, regarded as continuous linear functionals on L^∞, are called **normal functionals**. All functionals from $(L^\infty)^*$ for which the normal part is 0 are called **singular functionals**.

So we can speak out our question/problem in the following terms:

We search a positive, linear, faithful, normal functional $\phi \in L^{\infty *}$ such that $\phi(\mathcal{H}) = 0$ (where $\mathcal{H}$ is a given subspace $\mathcal{H} \subseteq L^\infty$).

A positive linear functional ϕ with $\phi(1_\Omega) = 1$ is also called a **state**.

Hence our title: "Subspaces annihilated by faithful normal states in abelian algebras".

So if one wants to solve the problem, one has to face the following problems. The functional ϕ that should annihilate the subspace $\mathcal{H}$ has to be:

- positive
- faithful
- normal

It is really hard to "make" a functional faithful and normal. Remember the criterion of **Kreps/Yan**, Theorem 1.2.2. They take the $\sigma(L^\infty, L^1)$-closure in L^∞. This topology is just the one on L^∞ which is induced by the normal functionals, i.e. by L^1.

Taking this topology, instead of lets say the weak topology $\sigma(L^\infty, L^{\infty *})$ or the norm topology $\|.\|_{L^\infty}$, guaranties that the annihilating functional is *normal*.

But they not only consider the space $\mathcal{H}$ itself, but the huge set $\mathcal{H} - L^\infty_+$. Considering this set, instead of lets say just $\mathcal{H}$, guarantees that the annihilating functional is *positive*.

The *faithfulness* they guarantee by a good argument which, beside, works only for normal functionals.

Note that the criterion of Kreps and Yan makes things not dramatically easier. One must consider the closure of $\mathcal{H} - L^\infty_+$ in the topology induced by the normal functionals. Thus the whole space L^1 must be taken into account and considered to get the closure. Hence, formulated exaggerated, we could just as well test each functionals of L^1 wether it is suitable or not.

1.2.2 Our concept

The proof of our result was visually motivated in the following way.

We are given the vector space L^∞ of which we think simply as the $\mathbb{R}^2$. The vector space L^∞ we decompose as a direct sum of the 1-dimensional subspace $\mathrm{span}(\mathbf{1})$ which consists of the constant functions, and the space E which let be consist of the functions f with $\int f dP = 0$.

$$L^\infty = \mathrm{span}(\mathbf{1}) \oplus \{ f \in L^\infty \mid \int f dP = 0 \}$$

$$= \mathrm{span}(\mathbf{1}) \oplus E.$$

Indeed, each vector $f \in L^\infty$ can be uniquely written as

$$f = \int f dP\mathbf{1} + \Big(f - \int f dP\mathbf{1}\Big),$$

where the first function

$$\int f dP\mathbf{1}$$

is a constant function and the second function $f - \int f dP\mathbf{1}$ has integral 0,

$$\int \Big(f - \int f dP\mathbf{1}\Big) dP = \int f dP - \int f dP \int \mathbf{1} dP = 0.$$

If we return to the simplification $\mathbb{R}^2 \equiv L^\infty$, we identify the 1-dimensional space $\mathrm{span}(\mathbf{1})$ with the Y-axis in $\mathbb{R}^2$ and the space E with the X-axis.

In $\mathbb{R}^2$ we paint the cone of positive functions L^∞_+ of L^∞.

See the subsequent Figure 1.1.

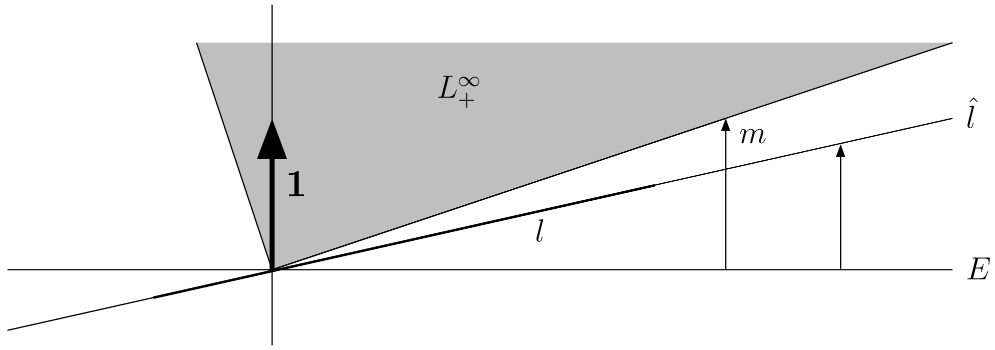

Figure 1.1: **This image shows the idea of our proof. The gray cone represents the positive functions from L^∞. The piece of line l represents the subspace $\mathcal{H}$. The line $\hat{l}$ represents the Hahn-Banach extension of l. The sublinear functional m is the distance to the surface of the cone.**

Here the subspace $\mathcal{H} \subseteq L^\infty$ is drawn as line l. We represent $\mathcal{H}$ in our coordinate system $\operatorname{span}(\mathbf{1}) \oplus E$,

$$\mathcal{H} = \{\, \alpha\mathbf{1} + x \mid x \in E, \alpha \in \mathbb{R}, \alpha\mathbf{1} + x \in \mathcal{H} \,\}.$$

Assume there are two points of the form $\alpha\mathbf{1}+x$ and $\beta\mathbf{1}+x$ ($\alpha, \beta \in \mathbb{R}$, $x \in E$) in the space $\mathcal{H}$. Then we would have

$$\alpha\mathbf{1} + x - (\beta\mathbf{1} + x) = (\alpha - \beta)\mathbf{1} \in \mathcal{H}.$$

If $\alpha \neq \beta$, then $\mathbf{1} \in \mathcal{H}$ and there is no chance to find a faithful probability measure that annihilates $\mathcal{H}$. So we can exclude that case and assume that $\alpha = \beta$. Thus with $\mathcal{H}$ we can associate a linear function

$$l : E_0 \to \mathbb{R}$$

such that $E_0 \subseteq E$ and $\mathcal{H}$ is nothing else as the graph of the function l,

$$\mathcal{H} = \{\, l(x)\mathbf{1} + x \mid x \in E_0 \,\}.$$

In the Figure 1.1 we have drawn a sublinear functional

$$m : E \to \mathbb{R}$$

which is nothing else as the distance $m(x)$ from a given point $x \in E$ in the direction $\mathbf{1}$ to the first point of a cone element. It turns out that

$$m(x) = -\inf(x),$$

where inf is the essential infimum in L^∞.

We search a functional that annihilates our subspace $\mathcal{H}$. But each linear functional can be represented as hyperplane which is defined as the kernel of the functional. So we search a hyperspace $\hat{\mathcal{H}}$ in L^∞ such that $\mathcal{H} \subseteq \hat{\mathcal{H}}$, because this hyperspace gives a linear functional $\rho : L^\infty \to \mathbb{R}$ with

$$\mathcal{H} \subseteq \hat{\mathcal{H}} = \ker(\rho).$$

In order that the sought annihilating functional ρ defines a measure equivalent to P it is necessary that its kernel does not contain a positive function of L^∞, so the sought hyperplane $\hat{\mathcal{H}}$ must not intersect the cone L^∞_+.

Now since $\mathcal{H} \cap L^\infty_+ = \emptyset$ we have

$$l(x) \leq m(x) \qquad \forall x \in E_0.$$

Compare our Figure 1.1. If we extend $l : E_0 \to \mathbb{R}$ to a linear map $\hat{l} : E \to \mathbb{R}$ via Hahn-Banach (since the sublinear functional $m : E \to \mathbb{R}$ majorizes l) such that

$$\hat{l}(x) \leq m(x) \qquad \forall x \in E,$$

then the graph of l gives a hyperplane in L^∞ which defines a *positive* functional ρ that annihilates $\mathcal{H}$. But it need not be *faithful.*

Indeed the hyperplane can touch the positive cone L^∞_+ so that $\rho(f) = 0$ for some nonzero positive function $f \in L^\infty_+$.

In order to avoid that the hyperplane touches the positive cone, in our work we extend the linear function $l : E_0 \to \mathbb{R}$ via Hahn-Banach to a linear functional $\hat{l} : E \to \mathbb{R}$ which is majorized by the sublinear functional cm for some given real constant $0 \leq c < 1$,

$$\hat{l} \leq cm.$$

Clearly here we must assume that

$$l(x) \leq cm(x) \qquad \forall x \in E,$$

so this is a restriction on the subspace $\mathcal{H}$ which is to annihilate.

Now our found functional ρ is *positive* and *faithful* since due to $\hat{l} \leq cm < m$ the graph of $\hat{l}$ can not touch the positive cone L^∞_+ (except in 0).

But we are not done! The functional ρ need not be *normal*. So we have a (necessarily bounded) functional $\rho : L^\infty \to \mathbb{R}$ but there is no evidence that we have indeed

$$\rho \in L^1(P) \subseteq (L^\infty)^*.$$

Some computations, which are visually not obvious to us, reveal the following: If in addition we have a real constant $C > 0$ such that

$$\hat{l}(x) \leq Cm(-x) \qquad \forall x \in E,$$

then the functional ρ associated to $\hat{l}$ is *normal*.

Thus we also claim

$$l(x) \leq Cm(-x) \qquad \forall x \in E_0.$$

Indeed in order we can apply Hahn-Banach's theorem we need a sublinear functional

$$n : E \to \mathbb{R}$$

such that

$$n(x) \leq cm(x)$$

and

$$n(x) \leq Cm(-x).$$

Such a functional can be easily defined and we claim

$$l \leq n.$$

Then Hahn-Banach's theorem gives a *faithful, normal, positive* functional $\rho : L^\infty_+ \to \mathbb{R}$ such that $\rho(\mathcal{H}) = 0$.

It turns out that the condition $l \leq n$ is equivalent to the existence of a linear functional ρ that annihilates $\mathcal{H}$ and such that

$$d\rho \leq \int .dP \leq D\rho$$

for some constants d, D which depend on c, C. If one chooses here

$$0 < d \leq 1 \leq D,$$

it is quite obvious that ρ is faithful, positive and normal.

1.2.3 The results

In the last section we have found some sublinear functional $n : E \to \mathbb{R}$ which we use as a criterion for the given subspace $\mathcal{H}$ via

$$l \leq n.$$

The explicit formula of n is

$$n(x) = \inf_{y \in L^\infty} \Big(C \sup [x + y - \int (x+y) dP\mathbf{1}] + c \sup [y - \int y dP\mathbf{1}] \Big).$$

Here sup symbols the essential supremum in L^∞.

This explicit formula of the sublinear functional n can be greatly optimized which is the longest proof, Lemma 3.2.2, appearing in the chapter.

We obtain (see Theorem 3.2.3 and page 72)

Theorem 1.2.3 *Let $(\Omega, \mathcal{F}, P)$ be a probability space,*
$\mathcal{H} \subseteq L^\infty := L^\infty(\Omega, \mathcal{F}, P, \mathbb{R})$ a linear subspace and $0 < c \leq 1 \leq C$. The following conditions are equivalent:

(1) *For all $x \in \mathcal{H}$*

$$\int x dP \leq C^{-1} \inf \Big\{ \ (C-1)\, \alpha + (c-1)\, \beta \ \Big|$$

$$\int x\,dP \leq \alpha \leq \sup(x),\ \beta \leq 0,\ \int (x-\alpha) \vee \beta\,dP = 0 \Big\}.$$

(2) *There exists a probability measure Q, which is equivalent to P, such that*

$$\int \mathcal{H} dQ = 0 \qquad \textit{and} \qquad c \leq dQ/dP \leq C.$$

Indeed slightly more general we can show a similar result for abelian C^*-algebras A instead for L^∞.

This theorem is our most accessible/computable result. We give by far less computable, but sufficient+necessary criterions for the following cases:

There exists a P-equivalent measure Q which annihilates $\mathcal{H}$ with one of the following additional properties:

- $$0 < c \leq dQ/dP \leq C < \infty.$$

  (This is once again the above Theorem 1.2.3.)

- $$dQ/dP \leq C < \infty.$$

- $$dQ/dP \in L^q \quad \text{and} \quad 0 < c \leq dQ/dP.$$

  Here $1 < q \leq \infty$.

- $$dQ/dP \in L^q.$$

  Here again $1 < q \leq \infty$.

Chapter 2

Doob-Meyer-decomposition of Hilbert-space valued functions

2.1 Introduction

(1) We give here a new proof of the Doob-Meyer-decomposition which is rather quick and elementary. It is more general in some aspects, and weaker in other aspects if compared to other approaches. The functions are defined on a totally ordered set with image in a Hilbert space. (2) We also give a second variant of the Doob-Meyer-decomposition; it is more specialized. (3) We apply (1) by reproving the Doléans-measure in a special setting and demonstrate that the stochastic integral could be defined on more general totally ordered time scales then $\mathbb{R}$.

In **chapter 2.2** we show a variant of the Doob-Meyer-decomposition. We decompose a certain class of Hilbert space $\mathcal{H}$ valued functions $f : \mathcal{R} \to \mathcal{H}$ on a totally ordered set $\mathcal{R}$ into a martingale and a predictable function. Positive, $L^2(\Omega)$- valued submartingales fall into this class. The kind of proof is new to our knowledge.

In **chapter 2.3** we show an analogous decomposition *in* $L^2(\mathcal{R}, \mu, \mathcal{H})$ of a certain class of functions $X \in L^2(\mathcal{R}, \mu, \mathcal{H})$, where $\mathcal{R}$ is the time scale, μ is a positive finite measure on it and $\mathcal{H}$ is Hilbert. Though we formulate $\mathcal{R}$ more generally as usual, one may think of $\mathcal{R}$ as $[0, 1]$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Q}$, $\mathbb{N}$, etc.

The proof given in chapter 2.3 is considerably more work than the proof given in chapter 2.2.

In **chapter 2.4** we apply the result of chapter 2.2 to obtain a proof of the Doléans-measure for a positive submartingale for a more general time scale $\mathcal{R}$ than $\mathbb{R}$.

We also demonstrate then, that a stochastic integral could be defined for this time scale.

2.2 Doob-Meyer-decompostion of $\mathcal{H}$-valued functions

We give here a rather quick Doob-Meyer-decomposition of a certain class of adapted functions on an arbitrary totally ordered set. Also compare [13], [15], [14], [5].

Let us be given a Hilbert space $\mathcal{H}$, a totally ordered set $\mathcal{R}$ and an increasing family $(P_t)_{t\in\mathcal{R}}$ of projections $P_t \in B(\mathcal{H})$, i.e. $\forall s,t \in \mathcal{R}:\ P_tP_t = P_t,\ P_t^* = P_t,\ \ s \leq t \Rightarrow P_sP_t = P_tP_s = P_s$.

With $(P_t)_{t\in\mathcal{R}}$ we associate a second increasing family of projections $(L_t)_{t\in\mathcal{R}}$ defined as

$$L_t = \bigvee_{s<t} P_s \qquad \text{i.e. the projection onto } \overline{\bigcup_{s<t} \mathrm{Im}(P_s)}.$$

We call a function $X : \mathcal{R} \to \mathcal{H}$ **adapted** if $P_tX_t = X_t$ for all $t \in \mathcal{R}$, **predictable** if $L_tX_t = X_t$ for all $t \in \mathcal{R}$, a **martingale** if $P_sX_t = X_s$ for all $s \leq t$ in $\mathcal{R}$.

Lemma 2.2.1 Fundamental lemma of the Doob-Meyer-decomposition

Let $\mathcal{H}$ be a Hilbert space. Let $(P_k)_{k\geq 0}$ be an increasing sequence of projections $P_k \in B(\mathcal{H})$. Let $X : \mathbb{N} \to \mathcal{H}$ be an adapted function, i.e. $\forall k \geq 1 :$ $P_kX_k = X_k$.

Then for the Doob-decomposition

$$A_n := \sum_{k=1}^{n} P_{k-1}(X_k - X_{k-1}), \qquad M_n := X_n - A_n \qquad (n \geq 0)$$

(thus $\forall n \geq 1 : X_n = M_n + A_n$ *and* $A_n = P_{n-1}A_n$ *and* $\forall k \leq n \, : \, M_k = P_k M_n$*) we have*

$$\|A_n\|^2 = 2\,Re < X_n, A_n > - \sum_{k=1}^{n} q(k) \qquad \textit{and} \qquad \|M_n\|^2 = \|X_n\|^2 - \sum_{k=1}^{n} q(k),$$

where $q(k) := \|P_{k-1}X_k\|^2 - \|X_{k-1}\|^2$.

Proof. Abbreviate $\Delta X_k := X_k - X_{k-1}$.

$$\begin{aligned}
\|A_n\|^2 &= < \sum_{k=1}^{n} P_{k-1}\Delta X_k, \sum_{i=1}^{n} P_{i-1}\Delta X_i > \\
&= 2\mathrm{Re} \sum_{1\leq i<k\leq n} < P_{k-1}\Delta X_k, P_{i-1}\Delta X_i > + \sum_{k=1}^{n} < P_{k-1}\Delta X_k, P_{k-1}\Delta X_k > .
\end{aligned}$$

Now we look at the first sum

$$\begin{aligned}
\beta &:= \sum_{1\leq i<k\leq n} < P_{k-1}\Delta X_k, P_{i-1}\Delta X_i > = \sum_{1\leq i<k\leq n} < \Delta X_k, P_{i-1}\Delta X_i > \\
&= \sum_{k=1}^{n} < \Delta X_k, \sum_{i=1}^{k-1} P_{i-1}\Delta X_i > \\
&= \sum_{k=1}^{n} (< X_k, A_{k-1} > - < X_{k-1}, A_{k-1} >) \\
&= \sum_{k=1}^{n} (< X_k, A_k - P_{k-1}\Delta X_k > - < X_{k-1}, A_{k-1} >) \\
&= \sum_{k-1}^{n} (< X_k, A_k > - < X_{k-1}, A_{k-1} >) - \sum_{k=1}^{n} < X_k, P_{k-1}\Delta X_k > \\
&= < X_n, A_n > - \sum_{k=1}^{n} < X_k, P_{k-1}\Delta X_k > .
\end{aligned}$$

Thus we get

$$\begin{aligned}
\|A_n\|^2 &= 2\mathrm{Re}\,\beta + \sum_{k=1}^{n} < P_k\Delta X_k, P_{k-1}\Delta X_k > = 2\mathrm{Re} < X_n, A_n > \\
&\quad + \sum_{k=1}^{n}(- < X_k, P_{k-1}\Delta X_k > - < P_{k-1}\Delta X_k, X_k > \\
&\quad + < P_{k-1}\Delta X_k, \Delta X_k >).
\end{aligned}$$

Expanding here the ΔX_k to $X_k - X_{k-1}$ we get

$$\begin{aligned}
\|A_n\|^2 &= 2\mathrm{Re} < X_n, A_n > \\
&\quad - \sum_{k=1}^{n}(< X_k, P_{k-1}X_k > - < P_{k-1}X_{k-1}, X_{k-1} >) \\
&= 2\mathrm{Re} < X_n, A_n > - \sum_{k=1}^{n} q(k).
\end{aligned}$$

And immediately we get the result

$$\|M_n\|^2 =< X_n - A_n, X_n - A_n >= \|X_n\|^2 - \sum_{k-1}^{n} q(k). \ \Box$$

Now consider as an example some probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_n)_{n\geq 1}$. If $X : \mathbb{N} \to L^2(\Omega, \mathcal{F}, P)$ is a positive submartingale, i.e. $0 \leq X_k \leq E(X_n|\mathcal{F}_k)$ for $k \leq n$, and setting P_n as the projection in $B(L^2(\Omega, \mathcal{F}, P))$ onto $L^2(\Omega, \mathcal{F}_n, P)$ and considering the fact $E(X_n|\mathcal{F}_k) = P_k X_n$ (e.g. [8] chapter 6), then $\|X_k\|^2 \leq \|P_k X_n\|^2$.

Thus $\sum_{k=1}^{n} q(k) \geq 0$, where q is defined as in the previous lemma. This estimate is then applicable on the next theorem. Slightly more generally, if X is a submartingale with $X \geq c$ uniformly for some constant c, then $X - c \geq 0$ is a positive submartingale, such that the next theorem is also applicable. (cf. also [13] Prop.1 and Th.2)

Theorem 2.2.2 Doob-Meyer-decomposition variant 1

Let $\mathcal{R}$ be a totally ordered set. Let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ an increasing family of projections in $B(\mathcal{H})$. Let $X : \mathcal{R} \to \mathcal{H}$ an adapted *function. For all $t \in \mathcal{R}$ let exist a constant $C_t \in \mathcal{R}$ such that for all $n \geq 1$ and all $t_0 < \ldots < t_n = t$ in $\mathcal{R}$ we have*

$$\sum_{k=1}^{n} \|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2 \geq C_t. \tag{2.1}$$

Then we find adapted functions $M, A : \mathcal{R} \to \mathcal{H}$ such that $X = M + A$ and M is a martingale and A is predictable.

If $\mathcal{H} = L^2(\Omega)$ and X is a positive submartingale then A can be chosen increasing (i.e. $A_s \leq A_t$ for all $s \leq t$ in $\mathcal{R}$).

Proof. Consider the set of finite subsets of $\mathcal{R}$, i.e.

$$\phi := \{\, u \subseteq \mathcal{R} \mid \operatorname{card}(u) < \infty \,\}.$$

Then consider the filter basis B consisting of all end pieces of ϕ, where we think of ϕ being ordered under the set inclusion, i.e.

$$B = \{\, \alpha \subseteq \phi \mid \exists u_0 \in \phi : \alpha = \{\, u \mid u \supseteq u_0 \,\}\,\}.$$

Complete this filter basis to an ultrafilter $\mathcal{U}$ on ϕ. For fixed $t \in \mathcal{R}$ set

$$A_t : \phi \to \mathcal{H} \ : \ A_t(u) = 0 \text{ for } t \notin u \quad \text{and}$$

$$A_t(u) = \sum_{k=1}^{n} P_{t_{k-1}}(X_{t_k} - X_{t_{k-1}}) \text{ for } u = \{t_0 < \ldots < t_n = t < t_{n+1} < \ldots < t_m\}$$

Set $M_t : \phi \to \mathcal{H} : M_t(u) = X_t - A_t(u)$. Using the fundamental lemma 2.2.1 we get $\|M_t(u)\|^2 = \|X_t\|^2$ for $t \notin u$, and for $t \in u$ we have

$$\|M_t(u)\|^2 = \|X_t\|^2 - \sum_{k=1}^{n} (\|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2) \leq \|X_t\|^2 - C_t.$$

Thus $\|A_t(u)\| = \|X_t - M_t(u)\| \leq \|X_t\| + \sqrt{\|X_t\|^2 - C_t}$. Note that the unit ball is compact under the weak topology in a Hilbert space. So we obtain weak limits along the ultrafilter, i.e. set (here wlim denotes the weak limit)

$$M : \mathcal{R} \to \mathcal{H} : M_t = \operatorname{wlim}_{\mathcal{U}} M_t(u)$$

and analogously define A. Note that the limits M_t respectively A_t are indeed in the subspace $\mathrm{Im}(P_t)$ respectively $\mathrm{Im}(L_t)$, so M respectively A are adapted respectively predictable. Now we get

$$X_t = \mathrm{wlim}_{\mathcal{U}} X_t - A_t(u) + A_t(u) = \mathrm{wlim}_{\mathcal{U}} M_t(u) + \mathrm{wlim}_{\mathcal{U}} A_t(u) = M_t + A_t.$$

Further more fix $s \leq t$ in $\mathcal{R}$. Note that for fixed $u \in \phi$ with $\{s, t\} \subseteq u$ we have $P_s M_t(u) = M_s(u)$. So we get

$$P_s M_t = P_s(\mathrm{wlim}_{\mathcal{U}} M_t(u)) = \mathrm{wlim}_{\mathcal{U}} P_s M_t(u) = \mathrm{wlim}_{\mathcal{U}} M_s(u) = M_s,$$

i.e. M is a martingale. Now let $h = \mathbb{C}$ and X be a submartingale, i.e. $P_s(X_t - X_s) \geq 0$ for all $s \leq t$ in $\mathcal{R}$. So we have $A_t(u) - A_s(u) \geq 0$ for all $s \leq t$ and all $u \supseteq \{s, t\}$. Thus for all $0 \leq f \in L^2(\Omega)$ we get

$$< A_t - A_s, f >= \mathrm{wlim}_{\mathcal{U}} < A_t(u) - A_s(u), f > \geq 0,$$

i.e. $A_t - A_s \geq 0$. $\square$

Examples. Note that not all processes $X = M + A$, M a martingale, A predictable, fulfill the sufficient condition of theorem 2.2.2. Indeed, e.g., set $\mathcal{H} = \mathbb{C}$, $\mathcal{R} = (0, 1)$ and $P_s = I$ for all $s \in (0, 1)$. Then all processes are predictable, but $X(s) := 1/s$ does not satisfy the condition in 2.2.2, indeed (2.1) becomes

$$\sum |X(t_k)|^2 - |X(t_{k-1})|^2 = |X(t_n)|^2 - |X(t_0)|^2 = 1/t_n^2 - 1/t_0^2 \to -\infty. \quad (2.2)$$

Further $X(s) := 1_{\mathbb{Q}}(s)$ fulfills the condition of 2.2.2 and shows that the resulting predictable function need not have bounded variation, as the martingale must be constant with respect to the given (constant) filtration.

2.3 Doob-Meyer-decomposition in $L^2(\mathcal{R}, \mu, \mathcal{H})$

Using the fundamental lemma 2.2.1, in this chapter we give a Doob-Meyer-decomposition of a certain class of adapted functions *in* the Hilbert space

$L^2(\mathcal{R}, \mu, \mathcal{H})$, where the time scale $\mathcal{R}$ is endowed with a finite measure μ.

For a totally ordered set $\mathcal{R}$ the **order topology** is induced by the open sets $\{ s \in \mathcal{R} \mid t < s \}$ and $\{ s \in \mathcal{R} \mid s < t \}$ for all $t \in \mathcal{R}$. Thus these "unbounded" intervals together with all finite open intervals $(s,t) \subseteq \mathcal{R}$ form a basis of the topology. $t \in \mathcal{R}$ is a **successor** of $s \in \mathcal{R}$ if $s < t$ and the open interval (s,t) is empty. Here s is a **predecessor** of t.

We make $\mathcal{R}$ a measurable space by endowing it with the Borel-Algebra $\mathcal{B}$. Let us be given a positive measure μ on $\mathcal{R}$. In this chapter we will restrict ourself to the following conditions.

(I) $\quad \mathcal{R}$ **is separable**
and has countable many successor elements

(II) $\quad \mu([s,t]) < \infty \qquad \forall s < t \in \mathcal{R}.$

Note for example that any subset $\mathcal{R} \subseteq \mathbb{R}$ fulfills condition (I): indeed at first one constructs a countable subset $D \subseteq \mathcal{R}$ which is dense in $\mathcal{R}$ with respect to the trace topology of $\mathbb{R}$. Then D is also dense in the coarser order topology.

Also note that there exist separable totally ordered sets, which have not countable many successors (e.g. take $\mathbb{R} \times \{0,1\}$ with lexicographical order).

Maybe every $\mathcal{R}$ fulfilling (I) is a subset in $\mathbb{R}$. We have not put effort in that direction.

Assume $(I)-(II)$. We can assume that the dense countable subset $D \subseteq \mathcal{R}$ contains also all elements which are successors or predecessors, i.e.

$$\{ s \in \mathcal{R} \mid \exists x < s : (x,s) = \emptyset \text{ or } \exists x > s : (s,x) = \emptyset \} \ \subseteq \ D. \tag{2.3}$$

Thus we note that D is left and right dense, i.e.

$$\forall s < t \qquad (s,t] \cap D \neq \emptyset \quad \text{and} \quad [s,t) \cap D \neq \emptyset. \tag{2.4}$$

Furthermore each open subset $O \subseteq \mathcal{R}$ can be written as a countable union of open intervals; indeed if O does neither contain $\min(\mathcal{R})$ nor $\max(\mathcal{R})$ (if they exist), then O can be written as $O = \bigcup_{x \in O} (\alpha_x, \beta_x)$ where one chooses $\alpha_x, \beta_x \in D$ such that $\alpha_x < x < \beta_x$ and $(\alpha_x, \beta_x) \subseteq O$, so that O is obviously written as countable union of open intervals. If O contains min / max one has to add at most two open "unbounded" intervals.

We conclude that $\mathcal{B}$ is induced by the open intervals and that $\mathcal{R}$ is the countable union of intervals of the form $(s, t]$ and of $\{\min(\mathcal{R})\}$, if $\min(\mathcal{R})$ exists. Furthermore each single point $\{s\}$, being closed, is measurable. Thus $\mathcal{B}$ is induced by the algebra $\mathcal{A}$ of finite unions of sets $(s, t]$ $(s, t \in \mathcal{R})$ and $\{\min(\mathcal{R})\}$.

We conclude that the step functions of $\mathcal{A}$ are dense in $L^1(\mathcal{R}, \mu)$ due to a theorem of the theory of measure extension (e.g. [11] VI.6). Thus they are dense in $L^2(\mathcal{R}, \mu)$. (Indeed if ϕ is a step function of the algebra $\mathcal{A}$ and $A \in \mathcal{B}$ then $\|1_A - (\phi \wedge 1) \vee 0\|_{L^2}^2 \le \|1_A - (\phi \wedge 1) \vee 0\|_{L^1} \le \|1_A - \phi\|_{L^1}$ and the last number can be made arbitrarily small.)

Now let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t \in \mathcal{R}}$ be an increasing family of projections in $B(\mathcal{H})$. For a function $X : \mathcal{R} \to \mathcal{H}$ to be **adapted**, **predictable** or a **martingale**, we take the definition of chapter 2.2.

We define the **space of predictable functions** as the closure in $L^2(\mathcal{R}, \mu, \mathcal{H})$ of

$$\operatorname{span} \{ 1_{(s,t]} \, \xi \mid s, t \in \mathcal{R}, \ \xi \in \operatorname{Im}(P_s) \}.$$

In lemma 2.3.2 we will show that the space of predictable functions is exactly the set of predictable functions (μ-a.e.).

Recall the definition of the increasing family of projections $(L_t)_{t \in \mathcal{R}}$ in chapter 2.2.

$$L_t = \bigvee_{s<t} P_s \qquad \text{i.e. the projection onto } \overline{\bigcup_{s<t} \operatorname{Im}(P_s)}.$$

Analogously we define a increasing family $(R_t)_{t \in \mathcal{R}}$ of projections by

$$R_t = \bigwedge_{s>t} P_s \qquad \text{i.e. the projection onto } \bigcap_{s>t} \operatorname{Im}(P_s).$$

In our setting, R_t is explicitly computable by the following general

Remark 2.3.1 *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{R}$ a totally ordered set and $(P_t)_{t\in\mathcal{R}}$ an decreasing family of projections in $B(\mathcal{H})$. Then the projection*

$$P = \bigwedge_{t\in\mathcal{R}} P_t \in B(\mathcal{H}) \qquad \textit{onto} \qquad \bigcap_{t\in\mathcal{R}} \mathrm{Im}(P_t)$$

is the strong operator limit of the net $(P_t)_{t\in\mathcal{R}}$.

I.e. $\forall \xi \in \mathcal{H} : P\xi = \lim_{t\uparrow} P_t\xi$ (i.e. the limit of the net $(P_t\xi)_{t\in\mathcal{R}}$).

Proof. So fix $\xi \in \mathcal{H}$ and consider the uniformly norm bounded net $(P_t\xi)_{t\in\mathcal{R}}$. It exists a subnet $(P_i\xi)_{i\in I}$ converging weakly to $\eta \in \mathcal{H}$. For fixed $t \in \mathcal{R}$ the limit acts in $\mathrm{Im}(P_t)$. So $\eta \in \mathrm{Im}(P_t)$. Thus $\eta \in \bigcap_{t\in\mathcal{R}} \mathrm{Im}(P_t)$ and therefore $P\eta = \eta$. Thus

$$P\xi = \mathrm{wlim}_i\, PP_i\,\xi = P\,\mathrm{wlim}_i\, P_i\xi = P\eta = \eta.$$

Now fix $\varepsilon > 0$. Then we find $s \in \mathcal{R}$ such that $| < P_s\xi - \eta, \xi > | \leq \varepsilon$. Thus

$$\begin{aligned}
\forall t \geq s \;:\; \|(P_t - P)\xi\|^2 &= \|(P_t - P)(P_s - P)\xi\|^2 \leq \|(P_s - P)\xi\|^2 \\
&= < (P_s - P)\xi, (P_s - P)\xi > = < (P_s - P)\xi, \xi > \\
&= < P_s\xi - \eta, \xi > \leq \varepsilon. \;\square
\end{aligned}$$

Let $\mathcal{H}_s = \mathrm{Im}(P_s)$. Then L_t is the projection onto $\overline{\bigcup_{s<t}\mathcal{H}_s} = ((\bigcup_{s<t}\mathcal{H}_s)^\perp)^\perp = (\bigcap_{s<t}\mathcal{H}_s^\perp)^\perp$. Thus $I - L_t$ projects onto $\bigcap_{s<t}\mathcal{H}_s^\perp$. Note that $I - P_s$ projects onto $\mathcal{H}_s^\perp$. Thus by remark 2.3.1 $I - L_t$ is the strong operator limit of the decreasing net $(I - P_s)_{s<t} \downarrow$, i.e. $(I - L_t)\eta = \lim_{s\uparrow t}(I - P_s)\eta = I\eta - \lim_{s\uparrow t} P_s\eta$. Thus

$$L_t\eta = \lim_{s\uparrow t} P_s\eta \qquad \forall \eta \in \mathcal{H}. \tag{2.5}$$

A function $f : \mathcal{R} \to Y$ that maps into a topological space Y call **right continuous** in $t \in \mathcal{R}$, if the restriction $f|\{s \geq t\}$ is continuous in t with respect to the trace topology on $\{s \geq t\}$.

An equivalent definition is, that for any net $(t_i) \subseteq \{s \geq t\}$ with limit t in the topology of $\mathcal{R}$, the net $f(t_i)$ has limit $f(t)$.

Note that if t has a successor t', then f is always right continuous in t, as t is isolated to the right.

Define the **right limit** of f in t as the limit of the net $(f(s))_{s>t}$, if it exists, and we write $\lim_{s \to t, s>t} f(s)$ or $\lim_{s \downarrow t} f(s)$ for this limit. It is straight forward to check that f is right continuous in t iff t has a successor or $\lim_{s \downarrow t} f(s) = f(t)$.

The family of projections $(P_t)_{t \in \mathcal{R}} \uparrow$ call **right continuous**, if it is right continuous with respect to the strong operator topology in $B(\mathcal{H})$. Or, equivalently, if $\forall \xi \in \mathcal{H}$ the function $t \to P_t \xi$ is right continuous. Also equivalent is the assertion

$$P_t = R_t \qquad \forall t \text{ which have no successor.} \tag{2.6}$$

We remark that given a family of projections $(P_t)_{t \in \mathcal{R}}$ we can construct a right continuous one, say $(\tilde{P}_t)_{t \in \mathcal{R}}$, by setting $\tilde{P}_t = R_t$ if t has no successor and $\tilde{P}_t = P_t$ otherwise.

Lemma 2.3.2 *Let $\mathcal{R}$ be totally ordered with measure μ and assume $(I)-(II)$. Let $\mathcal{H}$ be Hilbert and $(P_t)_{t \in \mathcal{R}}$ an increasing family of projections in $B(\mathcal{H})$. Then the projection $P \in B(L^2(\mathcal{R}, \mu, \mathcal{H}))$ onto the predictable function space is L, i.e.*

$$P(X)(t) = L_t X_t \qquad X \in L^2(\mathcal{R}, \mu, \mathcal{H}),\ t \in \mathcal{R}$$

Proof. Choose $D \subseteq \mathcal{R}$ countable, dense in $\mathcal{R}$ and such that (2.3) fulfilled. Let $(\alpha, \beta] \subseteq \mathcal{R}$ be a finite interval. Consider $X = 1_{(\alpha,\beta]} \xi \in L^2(\mathcal{R}, \mu, \mathcal{H})$, i.e. X is a simple step function.

Let $D' = (\alpha, \beta) \cap D = \{d_1, d_2, d_3, ...\}$. For fixed $n \geq 1$ order the set $\{d_1, ..., d_n\}$ and obtain, lets say, $\{\alpha =: e_0 < e_1 < ... < e_n < e_{n+1} := \beta\}$. Set

$$X_n = \sum_{k=1}^{n+1} 1_{(e_{k-1}, e_k]} P_{e_{k-1}} \xi.$$

Obviously X_n is predictable. Now let $t \in (\alpha, \beta]$.

If t is the successor of an element s then $L_t = P_s$ and $\{s, t\} \subseteq \{\alpha, \beta\} \cup D'$. Thus for some m we have $\lim_n X_n(t) = X_m(t) = P_s \xi = L_t X(t)$.

If t is not a successor then $(s,t) \cap D' \neq \emptyset \ \forall s < t$. Thus recalling (2.5)

$$\lim_n X_n(t) = \lim_{s \uparrow t, s \in D'} P_s \xi = \lim_{s \uparrow t, s \in \mathcal{R}} P_s \xi = L_t \xi = L_t X(t).$$

Summing up we have $\lim_n X_n(t) = L_t X(t)$ for all $t \in \mathcal{R}$, thus LX is μ-measurable as a pointwise limit of step functions.

If X is a linear combination of step functions like above then $\|LX\|^2 = \int \|L_t X(t)\|^2 d\mu(t) \leq \|X\|^2$. Due to the assumptions $(I) - (II)$ this step functions are dense in L^2 and we can continuously extend the map L on $L^2(\mathcal{R}, \mu, \mathcal{H})$.

LX is predictable for X step, thus $\mathrm{Im}(L)$ remains in the closed space of predictable functions. On the other hand obviously L does not alter a predictable step function $1_{(s,t]}\xi$, $\xi \in \mathrm{Im}(P_s)$. Thus $\mathrm{Im}(L)$ is exactly the space of predictable processes. Obviously L is idempotent and $L^* = L$, i.e.

$$< Lf, g >= \int < L_t f(t), g(t) > d\mu(t)$$

$$= \int < f(t), L_t g(t) > d\mu(t) =< f, Lg > .\square$$

Lemma 2.3.3 *Let $\mathcal{R}$ be totally ordered with measure μ and assume $(I) - (II)$. Let $\mathcal{H}$ be Hilbert and $(P_t)_{t \in \mathcal{R}}$ be an increasing, right continuous family of projections in $B(\mathcal{H})$. Then the projection $P \in B(L^2(\mathcal{R}, \mu, \mathcal{H}))$ onto the adapted functions is*

$$P(X)(t) = P_t X_t \qquad X \in L^2(\mathcal{R}, \mu, \mathcal{H}),\ t \in \mathcal{R}. \tag{2.7}$$

Proof. We show that one can analogously to lemma 2.3.2 define a function

$$R \in B(L^2(\mathcal{R}, \mu, \mathcal{H})) \qquad R(X)(t) = R_t X_t.$$

The problem is the μ-measurability of the expression. To avoid double work we reverse the order of $\mathcal{R}$ and obtain the order relation $\tilde{<}$. Transform the projections P_t to $\tilde{P}_t := I - P_t \uparrow$. Now lemma 2.3.2 shows that $\tilde{L}_t : L^2 \to L^2$ is defined. But

$$\tilde{L}_t(X)(t) = \lim_{s \tilde{<} t} \tilde{P}_s X_t = \lim_{s > t} (I - P_s) X_t = X_t - R_t X_t.$$

Thus $R = I_{L^2(\mathcal{R},\mu,\mathcal{H})} - \tilde{L}$, thus $R \in B(L^2(\mathcal{R},\mu,\mathcal{H}))$. Now let $S \subseteq \mathcal{R}$ the (countable) set of elements which have a successor. Then using (2.6) we have $R_t = P_t \; \forall t \in S^C$. Thus for step functions $X = 1_{(\alpha,\beta]}\xi$ ($\alpha < \beta \in \mathcal{R}, \xi \in \mathcal{H}$) we get that

$$Y : \mathcal{R} \to \mathcal{H} \; : \; Y(t) = P_t X_t = 1_{S\cap(\alpha,\beta]}(t)\, P_t\xi + 1_{S^C\cap(\alpha,\beta]}(t)\, R_t\xi$$

is μ-measurable. Thus the map (2.7) is well behaved for step functions X and can be extended continuously on $L^2(\mathcal{R},\mu,\mathcal{H})$. It is now easy to check that (2.7) is the projection onto the adapted functions. $\square$

In the next theorem we show an integrable version of theorem 2.2.2. It gives a decomposition in $L^2(\mathcal{R},\mu,\mathcal{H})$ and we now use convergence in that space rather than in $\mathcal{H}$ for each point of time. The proof is divided into steps and we urgently remark that the length of step 1a and step 1b, where we estimate the norm of some step function Y_{α_n} ((2.9) and (2.11)), is mainly due to the generality of the measure μ; finiteness is our only restriction. E.g. if $\mathcal{R} = (0,1)$ and μ the Lebesgue measure the estimation is simple. Or e.g. if we assume the function X in L^∞, the proof of step 1a/1b reduces to an immediate and trivial estimate ($\|Y_{\alpha_n}\|^2_{L^2(\mathcal{R},\mu,\mathcal{H})} \le \mu(\mathcal{R})\, \|X\|^2_{L^\infty(\mathcal{R},\mu,\mathcal{H})}$, compare (2.9), (2.11)).

Theorem 2.3.4 Doob-Meyer-decomposition variant 2

Let $\mathcal{R}$ be a totally ordered set with finite measure μ and assume (I). Let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ an increasing, right continuous family of projections in $B(\mathcal{H})$. Let $X \in \mathfrak{L}^2(\mathcal{R},\mu,\mathcal{H})$ be adapted and right continuous w.r.t. the weak topology in $\mathcal{H}$ and let exist a constant $C \in \mathbb{R}$ such that $\forall n \ge 1$ $\forall t_0 < ... < t_n \in \mathcal{R}$

$$\sum_{k=1}^{n} \|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2 \ge C.$$

Then we find $M, A \in \mathfrak{L}^2(\mathcal{R},\mu,\mathcal{H})$ such that $X = M + A$ μ-a.e., such that M is a right continuous martingale μ-a.e. and such that A is predictable μ-a.e.

If $\mathcal{H} = L^2(\Omega)$ and X is a positive submartingale then A can be chosen increasing, i.e. $A_s \le A_t$ for all $s \le t$ in $\mathcal{R}\backslash Z$ for some μ-null set Z.

Proof. **STEP 1a.** Consider the function

$$f : \mathcal{R} \to \mathbb{R} : f(s) = \mu(\{ t \in \mathcal{R} \mid t \leq s \}).$$

Notice $f \uparrow$. Fix $\alpha < \beta$ in $\mathcal{R}$ and consider $I = f([\alpha, \beta])$. Let exist $u_0 < u_1 < u_2 < u_3 \in \mathbb{R}$ such that

$$[f(\alpha), f(\beta)] = [u_0, u_1] \cup (u_1, u_2] \cup (u_2, u_3].$$

Now assume

$$(u_1, u_2] \cap I = \emptyset \qquad \text{and let} \qquad J := \{ s \in \mathcal{R} \mid f(s) > u_2 \}.$$

Our aim here is to show (2.8) below. Notice $J \neq \emptyset$ and $J^C \neq \emptyset$. Consider the dense countable subset $D \subseteq \mathcal{R}$ fulfilling (2.3) and also recall (2.4). At first let us assume $\nexists \inf J$ in $\mathcal{R}$. Then neither J nor J^C can be consist of a single point. Thus for $x \neq y$ in J we have $[x, y) \cap D \neq \emptyset$, thus $J \cap D \neq \emptyset$. Analogously $J^C \cap D \neq \emptyset$. Set $A = \bigcap_{s \in J^C \cap D,\, t \in J \cap D} (s, t]$. D is countable, thus

$$\mu(A) = \inf_{s \in J^C \cap D,\, t \in J \cap D} f(t) - f(s) \geq u_2 - u_1 > 0.$$

But A is empty: assume $x \in A \cap J$ then $x > y \in J$ for some y, thus $[y, x) \cap D \neq \emptyset$ thus $x \notin A \cap J$, contradiction. Similarly $A \cap J^C = \emptyset$, otherwise $\exists \max J^C$ and thus $\exists \inf J$. Summing up $A = \emptyset$ but $\mu(A) > 0$, thus the assumption $\nexists \inf J$ was wrong.

So let $x = \inf J$. Assume $x \in J^C$. Then $|J| > 1$ thus $J \cap D \neq \emptyset$. Consider $A = \bigcap_{t \in J \cap D} (x, t]$. $A = \emptyset$, as $x = \max J^C$. But $\mu(A) = \inf f(t) - f(x) \geq u_2 - u_1 > 0$. Contradiction, thus $x \in J$.

If $J^C = \{y\}$ then $\mu(\{x\}) = \mu((y, x]) = f(x) - \sup f(J^C)$. Otherwise

$$f(x) - \sup f(J^C) = \inf\{ f(x) - f(s) \mid s \in J^C \} = \inf_{s \in J^C} \mu((s, x]) \geq \mu(\{x\})$$

$$= \mu(\{x\}) = \mu(\bigcap_{s \in J^C \cap D} (s, x]) = f(x) - \sup f(J^C \cap D) \geq f(x) - \sup f(J^C).$$

Additionally we remark $f(x) = \min(I \cap (u_2, u_3])$. Summing up we have shown

$$\exists x \in \mathcal{R} \qquad x = \min J \quad \text{and} \quad \mu(\{x\}) = f(x) - \sup(I \cap [u_0, u_1]). \tag{2.8}$$

STEP 1b. For $\alpha = \{\alpha_0 < ... < \alpha_N\} \subseteq \mathcal{R}$ set

$$Y_\alpha = \sum_{k=1}^{N} 1_{(\alpha_{k-1},\alpha_k]} X_{\alpha_k} \in L^2(\mathcal{R}, \mu, \mathcal{H}). \tag{2.9}$$

Note that $\|X(s)\|^2 + C \leq \|P_s X(t)\|^2 \leq \|X(t)\|^2$ for all $s \leq t$ in $\mathcal{R}$. So we get

$$\begin{aligned}
\|Y_\alpha\|^2 - \|X\|^2 &\leq \int_{(\alpha_0,\alpha_N]} \|Y_\alpha(s)\|^2 - \|X(s)\|^2 ds \\
&= \sum_{k=1}^{N} \int_{(\alpha_{k-1},\alpha_k]} \|X(\alpha_k)\|^2 - \|X(s)\|^2 ds \\
&\leq \sum_{k=1}^{N} \int_{(\alpha_{k-1},\alpha_k]} \|X(\alpha_k)\|^2 - \|X(\alpha_{k-1})\|^2 - C\, ds \\
&\leq \sum_{k=1}^{N} (f(\alpha_k) - f(\alpha_{k-1}))\, (\|X(\alpha_k)\|^2 - \|X(\alpha_{k-1})\|^2) \; + \; \mu(\mathcal{R})|C|.
\end{aligned} \tag{2.10}$$

Now decompose equidistantly the interval $[f(\alpha_0), f(\alpha_N)] = [u_0, u_1] \cup (u_1, u_2] \cup ... \cup (u_{n-1}, u_n]$, i.e. for some $\varepsilon \geq 0$, $u_k - u_{k-1} = \varepsilon$ for all $1 \leq k \leq n$. Construct a finite set $\beta \subseteq \mathcal{R}$ with $\alpha \subseteq \beta$ by adding points x_k to α for each k such that $f(x_k) \in (u_{k-1}, u_k]$, if possible. If for some k the set $f([\alpha_0, \alpha_N]) \cap (u_{k-1}, u_k]$ is empty then add the point x with nonzero point mass and $f(x) = \min(f([\alpha_0, \alpha_N]) \cap (u_k, u_n])$ that we have found in 'step 1a'. Let $\beta = \{\alpha_0 = \beta_0 < \beta_1 < ... < \beta_M = \alpha_N\}$. Set

$$\gamma_1 = \{\, 1 \leq k \leq M \mid f(\beta_k) - f(\beta_{k-1}) \leq 2\varepsilon \,\}.$$

Set $\gamma_2 = \{1, ..., M\} \backslash \gamma_1$. For all $k \in \gamma_2$, β_k must be one of the special points with nonzero point mass and taking into account the result of 'step 1a' about the value of this mass we get

$$\mu(\{\beta_k\}) + \varepsilon \geq f(\beta_k) - f(\beta_{k-1}) > 2\varepsilon.$$

Abbreviate

$$g(k) := \|X(\beta_k)\|^2 - \|X(\beta_{k-1})\|^2.$$

Set $\gamma_{1+} = \{\, k \in \gamma_1 \mid g(k) \geq 0 \,\}$ and $\gamma_{1-} = \gamma_1 \backslash \gamma_{1+}$. Analogously represent $\gamma_2 = \gamma_{2+} \cup \gamma_{2-}$. We are ready to estimate (continuing (2.10))

$$\|Y_\beta\|^2 - \|X\|^2 \leq \sum_{k\in\gamma_{1+}} 2\varepsilon\, g(k) + \sum_{k\in\gamma_{2+}} (\mu(\{\beta_k\}) + \varepsilon)\, g(k) + \sum_{k\in\gamma_{2-}} 2\varepsilon\, g(k) \; + \; \mu(\mathcal{R})|C|$$

$$= \; 2\varepsilon \sum_{k=1}^{M} g(k) \; - \; 2\varepsilon \sum_{k\in\gamma_{1-}} g(k) + \sum_{k\in\gamma_{2+}} (\mu(\{\beta_k\}) - \varepsilon)\, g(k) \; + \; \mu(\mathcal{R})|C|$$

$$\leq \; 2\varepsilon(\|X(\beta_M)\|^2 - \|X(\beta_0)\|^2) \; - 2\varepsilon \sum_{k\in\gamma_{1-}} C + \sum_{k\in\gamma_{2+}} \mu(\{\beta_k\})\|X(\beta_k)\|^2 \; + \; \mu(\mathcal{R})|C|$$

$$\leq \; 2\varepsilon\|X(\alpha_N)\|^2 \; + \; 2\varepsilon\,|\gamma_{1-}|\,|C| \; + \; \|X\|^2_{L^2(\mathcal{R},\mathcal{H})} \; + \; \mu(\mathcal{R})|C|.$$

Recall we have chosen an equidistant decomposition into n intervals of $[f(\alpha_0), f(\alpha_N)]$. Notice that $\varepsilon \leq \mu(\mathcal{R})/n$. Choose n big enough such that $2\varepsilon\|X(\alpha_N)\|^2 \leq 1$ and that $N \leq n$. Recall the situation when we had filled each interval $(u_{k-1}, u_k]$ to obtain β. There chose maximal only one new β_i for each interval. Thus $M \leq N + n \leq 2n$. Thus $2\varepsilon\,|\gamma_{1-}|\,|C| \leq 2\frac{\mu(\mathcal{R})}{n} 2n|C| = 4\mu(\mathcal{R})|C|$.

Thus for fixed decomposition α of the interval $[\alpha_0, \alpha_N]$ we find a finer $\beta \supseteq \alpha$ one such that

$$\|Y_\beta\|^2 - \|X\|^2 \leq 1 + 4\mu(\mathcal{R})|C| + \|X\|^2 + \mu(\mathcal{R})|C| =: \delta.$$

Now choose intervals $(a_1, b_1] \subseteq (a_2, b_2] \subseteq ...$ such that $\bigcup_{k=1}^\infty (a_k, b_k] = \mathcal{R}$. (We assume here $\mathcal{R}$ has no minimum. A slight addition by taking also $\{\min\}$ into account would assimilate the general case.) Choose a dense countable subset $D \subseteq \mathcal{R}$ fulfilling (2.3). $D = \{d_1, d_2, ...\}$. Define inductively: Start with $\alpha_1 := \{a_1, b_1\}$. If $\alpha_{n-1} \subseteq \mathcal{R}$ is a decomposition of $(a_{n-1}, b_{n-1}]$ then add a_n, b_n and $(a_n, b_n] \cap \{d_1, ...d_n\}$ to α_{n-1}. Choose a finer decomposition α_n of this new α_{n-1} such that $\|Y_{\alpha_n}\|^2 - \|X\|^2 \leq \delta$. Summing up we get

$$\forall n \geq 1 \quad \|Y_{\alpha_n}\|^2 \leq \|X\|^2 + \delta \qquad \text{and} \qquad D \subseteq \bigcup_{n=1}^\infty \alpha_n\,, \qquad \alpha_1 \subseteq \alpha_2 \subseteq ... \tag{2.11}$$

STEP 2. Fix $n \geq 1$. For our $\alpha_n = \{t_0 < ... < t_N\}$ consider the Doob-Meyer-decomposition to the process $X(t_0), ..., X(t_N)$ (see lemma 2.2.1) and we obtain $A(t_k)$ and $M(t_k)$ for $0 \leq k \leq N$, where M is the martingale part and A is the predictable part, i.e. $A(t_k) \in \mathrm{Im}(P_{t_{k-1}})$. Define adapted processes $M_n, A_n \in \mathcal{K} := L^2(\mathcal{R}, \mu, \mathcal{H})$

$$A_n = \sum_{k=1}^{N} 1_{(t_{k-1},t_k]} A(t_k) \qquad \text{and}$$

$$M_n(s) = P_s M(t_N) \qquad \forall s \in \mathcal{R}.$$

We remark that due to lemma 2.3.3 M_n is the projection of $1_{\mathcal{R}} M(t_N)$ onto the adapted functions, thus indeed $M_n \in \mathcal{K}$. Further note that $M_n(t_k) = P_{t_k} M(t_N) = M(t_k)$. Thus due to the fundamental lemma 2.2.1 and our assumption

$$\forall t_{k-1} < s \leq t_k \quad \|M_n(s)\|^2 \leq \|M_n(t_k)\|^2 \leq \|X(t_k)\|^2 - C = \|Y_{\alpha_n}(s)\|^2 - C \tag{2.12}$$

$$\forall s \geq t_N \quad \|M_n(s)\|^2 = \|M(t_N)\|^2 \leq \|X(t_N)\|^2 - C \leq \|X(s)\|^2 - 2C$$

We estimate (also recall $\|Y_{\alpha_n}\|^2 \leq \|X\|^2 + \delta$; note $M_n(t_0) = X(t_0)$)

$$\|M_n\|^2 = \int_{\mathcal{R}} \|M_n(s)\|^2 ds$$

$$\leq \mu(\mathcal{R})\|X(t_0)\|^2 + \int_{t_0}^{t_N} \|Y_{\alpha_n}(s)\|^2 - C\, ds + \int_{s>t_N} \|X(s)\|^2 - 2C\, ds$$

$$\leq \mu(\mathcal{R})(\|X(a_1)\|^2 - C) + \|X\|^2 + \delta + \mu(\mathcal{R})\,|C| + \|X\|^2 + 2\mu(\mathcal{R})\,|C|$$

Analogously we get $\forall t_{k-1} < s \leq t_k$ (recall (2.12))

$$\|A_n(s)\| = \|A(t_k)\| = \|X(t_k) - M(t_k)\|$$

$$\leq \|Y_{\alpha_n}(s)\| + \sqrt{\|Y_{\alpha_n}(s)\|^2 - C} \leq 2\|Y_{\alpha_n}(s)\| + \sqrt{|C|}.$$

Thus

$$\|A_n\| \leq 2\sqrt{\|X\| + \delta} + \sqrt{\mu(\mathcal{R})}\sqrt{|C|}.$$

So $(A_n, M_n)_{n\geq 1}$ defines a bounded sequences in $\mathcal{K} \oplus \mathcal{K}$ and we obtain a weakly convergent subsequence $(A_{n_k}, M_{n_k})_{k\geq 1}$, as it is well known for Hilbert spaces. The limit be (A, M).

STEP 3. We show that $X = M + A$. Consider a finite interval $\beta \subseteq \mathcal{R}$. Choose m such that $\beta \subseteq \alpha_n$ for all $n \geq m$. Let $t(n,s) = \min\{\, t \in \alpha_n \mid s \leq t \,\}$. Let $\xi \in \mathcal{H}$. We compute

$$< M_n + A_n - X, 1_\beta \xi >= \int_\beta < M_n(s) + A_n(s) - X(s), \xi > ds$$

$$= \int_\beta < P_s M_n(t(n,s)) + P_s A_n(t(n,s)) - X(s), \xi > ds$$

$$= \int_\beta < P_s X(t(n,s)) - X(s), \xi > ds$$

Recall that $\alpha_1 \subseteq \alpha_2 \subseteq \ldots$ and $D \subseteq \bigcup_{k \geq 1} \alpha_k$. We have

$$| < P_s X(t(n,s)) - X(s), \xi > | \leq (\|X(t(n,s))\| + \|X(s)\|) \, \|\xi\|$$

$$\leq (\|X(t(m,s))\| - C + \|X(s)\|) \, \|\xi\| = (\|Y_{\alpha_m}(s)\| - C + \|X(s)\|) \, \|\xi\|.$$

This function is in $L^1(\mathcal{R}, \mu)$. We therefore can apply dominated convergence to get

$$< M + A - X, 1_\beta \xi >= \lim_k < M_{n_k} + A_{n_k} - X, 1_\beta \xi >$$

$$= \int_\beta \lim_k < X(t(n_k,s)) - X(s), P_s \xi > ds = 0,$$

for the last equality using the weakly right continuity of X. We have shown that $X = M + A$ μ-a.e.

Now note that M_n is a martingale and A_n is predictable for all $n \geq 1$. So the weak limits M and A must be a martingale respectively predictable as the space of martingales respectively space of predictable processes is norm-closed. Indeed the space of predictable functions is closed by definition. The $\mathcal{L}^2$-space of martingales define as

$$m = \{\, M \in \mathcal{L}^2(\mathcal{R}, \mu, \mathcal{H}) \mid \exists \text{ nullset } Z \subseteq \mathcal{R} \;:\; M|_{\mathcal{R}-Z} \text{ is a martingale} \,\}.$$

Its L^2-space is closed: let a sequence $M_n \to M$ in L^2-seminorm where $M_n \in m$, $M \in \mathcal{L}^2$. Then $M_{n_k}(s) \to M(s)$ for a subsequence for almost all $s \in \mathcal{R}$. Thus clearly $P_s M(t) = P_s \lim_k M_{n_k}(t) = \lim_k P_s M_{n_k}(t) = M(s)$ for all $s, t \in \mathcal{R} - Z$ for some null set Z, so $M \in m$.

We extend the 'carrier' $\Gamma \subseteq \mathcal{R}$, $\mu(\Gamma^C) = 0$, of M, where it is a martingale, by setting for all $s \in \mathcal{R}$, $M_s := P_s M_t$ for some $t \geq s$ (axiom of choice) if such t exists. The new carrier Γ_2 is so big, such that $\forall x \notin \Gamma_2 : x > \Gamma_2$. If $\exists \max(\Gamma_2)$ then we can (obviously) extend Γ_2 such that $\Gamma_2 = \mathcal{R}$. Anyway, M is now right continuous on Γ_2: fix $s \in \Gamma_2$, let $S > s$, $S \in \Gamma_2$. Then using the r.c. of the family of projections, $\lim_{t \downarrow s} M_t = \lim_{t \downarrow s} P_t M_S = P_s M_S = M_s$.

For the last statement, when X is a submartingale and the family of projections $(P_t)_{t \in \mathcal{R}}$ is (naturally) induced by a filtration $(\mathcal{F}_t)_{t \in \mathcal{R}}$, then A_n is increasing $\forall n \geq 1$. Recall that every norm-closed, convex set is closed under the weak topology. So A is increasing as the weak limit acts in the norm-closed (convex) cone of all increasing processes. Here the increasing processes define analogously as the space m and argue analogously for its closeness. $\square$

Remark. Again (as we also remarked below theorem 2.2.2) theorem 2.3.4 does not decompose all decomposable processes. Indeed, e.g., one could set $\mathcal{H} = \mathbb{C}$, $\mathcal{R} = (0, 1)$, μ the Lebesgue measure, $P_s = I$ for $s \in (0, 1)$, and $X(s) := s^{-1/4}$ (please compare (2.2)).

2.4 Application of the discrete Doob-Meyer-decomposition: Doleans-measure and integral associated to a martingale on a totally ordered set

For this chapter also compare [4] (Doléans-measure); and [10] or e.g. [12], [1] for stochastic integration. Also cf. [2] for the time scale to be $\mathbb{R}^2$.

In this chapter we are given a totally ordered set $\mathcal{R}$, a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t \in \mathcal{R}}$, i.e. an increasing family of σ-algebras in $\mathcal{F}$.

For the definition of right continuity and right limit, we refer to the definitions in chapter 2.3.

A totally ordered set $\mathcal{R}$ let us call **sequentially right complete** if every countable subset in $\mathcal{R}$ bounded from below has an infimum in $\mathcal{R}$, i.e.

$$\forall M \subseteq \mathcal{R} : \forall x \in \mathcal{R} : M \text{ countable } \wedge\ x \leq M \to \exists \inf M.$$

A function $f : \mathcal{R} \to Y$ into a topological space Y call **sequentially right continuous** if f is right continuous in all points of the set

$$\mathcal{S} := \{ s \in \mathcal{R} \mid \exists C \subseteq \mathcal{R} : C \text{ countable },\ s < C\,,\ s = \inf C \}. \tag{2.13}$$

The **predictable σ-algebra** $\mathcal{P}$ is defined as the σ-algebra in $\mathcal{R} \times \Omega$, generated by the sets

$$\{ (s,t] \times C \mid s,t \in \mathcal{R}\,,\ s < t\,,\ C \in \mathcal{F}_s \}.$$

Theorem 2.4.1 Doléans-measure

Let $\mathcal{R}$ be a totally ordered, sequentially right complete set. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$. Let $X : \mathcal{R} \to L^2(\Omega)$ a positive submartingale, such that $t \to E(X_t)$ is sequentially right continuous. Then

$$\mu(C \times (s,t]) = E(1_C\,(X_t - X_s)),$$

for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$, defines a measure on the predictable σ-algebra $\mathcal{P}$.

Proof. Consider the set algebra which consists of finite unions of elementary rectangles of the form $C \times (s,t]$ for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$. Due to the extension theorem of Hahn (e.g. [11] VI.7) it remains to show that μ is σ-additive on this algebra.

For a disjoint union $\bigcup_{k=1}^N r_k$ of elementary rectangles r_k the additively defined μ, i.e. $\mu(\bigcup_{k=1}^N r_k) = \sum_{k=1}^N \mu(r_k)$, is well defined. We skip the well-known and easy proof.

We now show that μ is σ-additive. Let

$$\bigcup_{k=1}^{\infty} C_k \times (s_k, t_k] = C \times (S,T]$$

a disjoint decomposition where $C_k \in \mathcal{F}_{s_k}$, $C \in \mathcal{F}_S$ and $s_k < t_k$, $S < T$ in $\mathcal{R}$.

We apply the Doob-Meyer-decomposition theorem 2.2.2 to the positive submartingale X on the interval $[S,T]$, i.e. we obtain adapted functions $M, A : [S,T] \to L^2(\Omega)$ such that $X = M + A$ on $[S,T]$, such that M is a martingale and such that A is increasing. Thus we can simplify the definition of μ: $(s < t,\ D \in \mathcal{F}_s)$

$$E(1_D(X_t - X_s)) = \int 1_D\, E(M_t + A_t - M_s - A_s|\mathcal{F}_s)dP = \int 1_D(A_t - A_s)dP.$$

Thus we have

$$\mu(D \times (s,t]) = \int 1_D(A_t - A_s)dP. \tag{2.14}$$

$E(X)$ was assumed to be sequentially right continuous thus

$$\lim_{t \downarrow s} \int |A_t - A_s|dP = \lim_{t \downarrow s} \int X_t - X_s dP = 0.$$

for all points $s \in \mathcal{S}$ defined by (2.13). Thus A is sequentially $L^1(\Omega)$-right continuous.

Now we reduce our 'time' scale to the countable index set

$$I := \{S, T, s_1, t_1, s_2, t_2, s_3, t_3, ...\}.$$

As $A_s \le A_t$ for all $s \le t$ we find a null set $Z \subseteq \Omega$ such that $A_s(\omega) \le A_t(\omega)$ $\forall \omega \in \Omega - Z$, $\forall s \le t \in I$. Fix $i \in I$. Assume $i = \inf\{\, s \in I \mid s > i \,\}$. Then $\inf_{s > i, s \in I} \|A_s - A_i\|_{L^1(\Omega)} = 0$ as A is L^1-right continuous in i. Thus $\lim_n \|A_{v_n} - A_i\|_{L^1(\Omega)} = 0$ for a sequence $v_n > i$, $v_n \in I$. Thus we find a subsequence (u_n) of (v_n) and a null set $Y \subseteq \Omega$ such that $\lim_n A_{u_n}(\omega) - A_i(\omega) = 0$ $\forall \omega \in \Omega - Y$. Especially $\forall \omega \in \Omega - (Y \cup Z)$ we get

$$0 \le \inf_{s > i, s \in I} A_s(\omega) - A_i(\omega) \le \inf_n A_{u_n}(\omega) - A_i(\omega) = 0.$$

As I is countable we find a null set $Z_2 \supseteq Z$ such that

$$\forall i \in I : \forall \omega \in \Omega - Z_2 : i = \inf\{\, s \in I \mid s > i \,\} \Rightarrow \inf_{s > i, s \in I} A_s(\omega) - A_i(\omega) = 0. \tag{2.15}$$

Now with monotone convergence and recalling (2.14) we get

$$\sum_{k=1}^{\infty} \mu(C_k \times (s_k, t_k]) = \int \sum_{k=1}^{\infty} 1_{C_k}(\omega)\,(A_{t_k}(\omega) - A_{s_k}(\omega))\,d\omega. \tag{2.16}$$

For fixed $\omega \in C - Z_2$ we consider the right sum. Abbreviate $A(s) := A_s(\omega)$. We drop all summands with $1_{C_k}(\omega) = 0$ and obtain a subsequence $(u_k, v_k)_k$ of $(s_k, t_k)_k$ such that $\bigcup_k (u_k, v_k] = (S, T]$ is a disjoint partition. Thus

$$\sum_{k=1}^{\infty} 1_{C_k}(\omega)(A(t_k) - A(s_k)) = \sum_{k=1}^{\infty} A(v_k) - A(u_k)$$

$$= \sum_k \lambda(\,(A(u_k), A(v_k)]\,) = \lambda(\bigcup_k (A(u_k), A(v_k)]\,) = \lambda(l),$$

λ being the Lebesgue measure, $l \subseteq \mathbb{R}$ being obviously defined.

Assume $l \neq (A(S), A(T)]$, then $\exists x \in (A(S), A(T)]\backslash l$. Set

$$J = \{u_1, v_1, u_2, v_2, ...\} \subseteq I \qquad \text{and} \qquad U = \{\, i \in J \mid A(i) \geq x \,\}.$$

Note $U \neq \emptyset$ as $T \in U$. $\exists \gamma = \inf U$ using the sequentially completeness of $\mathcal{R}$. $\gamma \in [S, T]$. At first look at the case $\gamma > S$. Then we find $n \geq 1$ with $\gamma \in (u_n, v_n]$. Then $v_n \leq U$. Thus $\gamma = v_n \in J$. Especially $u_n \notin U$. Thus $A(u_n) < x$. Anyway $\gamma \in J$. First case: $\gamma \in U$. Then $A(\gamma) \geq x$. Thus $\gamma \neq S$ as $A(S) < x$. But then $x \in (A(u_n), A(v_n)]$. Contradiction. Second case: $\gamma \notin U$. Then $U \subseteq \{\, i \in I \mid i > \gamma \,\} =: U_2$. Thus $\gamma \leq \inf U_2 \leq \inf U = \gamma$. And note if $\gamma < i \in I$ then $\exists u \in U : u \in (\gamma, i)$, so $A(i) \geq A(u)$. Thus recalling (3.2)

$$0 = \inf_{i > \gamma, i \in I} A(i) - A(\gamma) = \inf_{u \in U} A(u) - A(\gamma) \geq x - A(\gamma).$$

Thus $\gamma \in U$, contradiction. We conclude

$$\sum_{k=1}^{\infty} 1_{C_k}(\omega)(At_k(\omega) - A_{s_k}(\omega)) = 1_C(\omega)(A(T) - A(S)) \qquad \forall \omega \in \Omega - Z_2.$$

Thus $\sum_{k=1}^{\infty} \mu(C_k \times (s_k, t_k]) = \mu(C \times (S, T])$ with (2.16) $\square$

With the filtration we can naturally associate an increasing family of projections $(P_t)_{t\in\mathcal{R}}$, $P_t \in B(L^2(\Omega, \mathcal{F}, P))$, where P_t projects onto $L^2(\Omega, \mathcal{F}_t, P)$. For the definition of a right continuous such family, we refer to chapter 2.3.

Lemma 2.4.2 *Let $\mathcal{R}$ be a totally ordered set and $(\mathcal{F}_s)_{s\in\mathcal{R}}$ a filtration. Fix $t \in \mathcal{R}$. If the filtration is right continuous in t then the family $(P_s)_{s\in\mathcal{R}}$ is right continuous in t.*

If the filtration is augmented, i.e. each $\mathcal{F}_s$ contains the P-null-sets of $\mathcal{F}$, then we can reverse this conclusion.

Proof. Remark: if t has a successor element s we always say the filtration is right continuous in t as t is isolated to the right in the sense that $[t, s) = \{t\}$. Thus in that case both the filtration and the projections are automatically right continuous. Thus assume t has no successor. View $\mathcal{H}_s = L^2(\Omega, \mathcal{F}_s)$ as a closed subspace in $\mathcal{H} = L^2(\Omega, \mathcal{F})$. Consider the isometrically identic embedding $i : L^2(\Omega, \bigcap_{s>t} \mathcal{F}_s) \to \bigcap_{s>t} L^2(\Omega, \mathcal{F}_s)$. i is surjective: indeed, we can assume the filtration is augmented as it does not change the L^2-spaces; let $f \in \mathcal{H}$ be a $\mathcal{F}$-measurable representant of the right space. $[f] \in L^2(\mathcal{F}_s)$ for $s > t$, thus f is $\mathcal{F}_s$-measurable, thus f is $\bigcap_{s>t} \mathcal{F}_s$ measurable, thus $[f] \in L^2(\Omega, \bigcap_{s>t} \mathcal{F}_s)$.

Recall R_t is the projection onto $\bigcap_{s>t} \mathcal{H}_s$. Thus we get the filtration is right continuous in $t \iff \mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \implies \mathcal{H}_t = \bigcap_{s>t} \mathcal{H}_s \iff P_t = R_t$ (please also compare 2.3.1 and (2.6)) The only right arrow can be reversed if the filtration is augmented. □

Lemma 2.4.3 *Let $\mathcal{R}$ be a totally ordered, sequentially right complete set. Let $(\Omega, \mathcal{F}, P)$ be a probability space with sequentially right continuous filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$. Let h be a Hilbert space and $M : \mathcal{R} \to L^2(\Omega, h)$ be a martingale. Then*

$$\mu(C \times (s, t]) = \|1_C (M_t - M_s)\|^2_{L^2(\Omega,h)},$$

for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$, defines a measure on the predictable σ-algebra $\mathcal{P}$.

Proof. Please note that we have the well-known (and easy) identity

$$\|1_C\,(M_t - M_s)\|^2_{L^2(\Omega,h)} = \|1_C\,M_t\|^2 - \|1_C\,M_s\|^2 = E(1_C(X_t - X_s)),$$

where $X : \mathcal{R} \to L^1(\Omega)$ and $X_t(\omega) = \|M_t(\omega)\|^2_h$ and X is a positive submartingale.

We note that if the filtration is right continuous in a point t, a martingale automatically is L^2-right continuous there. Indeed lemma 2.4.2 shows that for some fixed $T > t$ in $\mathcal{R}$ we have $\lim_{s\downarrow t} M_s = \lim_{s\downarrow t} P_s M_T = P_t M_T = M_t$.

Thus if we assume that $M : \mathcal{R} \to L^4(\Omega, h)$, then we can directly apply theorem 2.4.1 on $X : \mathcal{R} \to L^2(\Omega)$.

For the general case we show that μ is σ-additive. Let $\bigcup_{k=1}^{\infty} C_k \times (s_k, t_k] = C \times (S, T]$ a disjoint decomposition where $C_k \in \mathcal{F}_{s_k}$, $C \in \mathcal{F}_S$ and $s_k < t_k$, $S < T$ in $\mathcal{R}$. Consider the projections $Q_k : L^2(\Omega, h) \to L^2(\Omega, h) : Q_k(\xi) = 1_{C_k}(P_{t_k} - P_{s_k})\xi$, where $k \geq 1$. Consider the projection $Q : L^2(\Omega, h) \to L^2(\Omega, h) : Q(\xi) = 1_C(P_T - P_S)\xi$. Note that the Q_k are pairwise orthogonal (i.e. $Q_k Q_i = 0$ for $k \neq i$) and all Q_k are smaller than Q (i.e. $Q_k Q = Q_k$).

For μ to be σ-additive we have to show $\sum_{k=1}^{\infty} \mu(C_k \times (s_k, t_k]) = \mu(C \times (S, T])$. Equivalently $\sum_{k=1}^{\infty} \|Q_k M_T\|^2 = \|Q M_T\|^2$. Equivalently $0 = \|Q M_T\|^2 - \sum_{k=1}^{\infty} \|Q_k M_T\|^2 = \|(Q - \sum_{k=1}^{\infty} Q_k) M_T\|^2$, where the convergence of $\sum_k Q_k$ is respect the strong-operator-topology (i.e. pointwise; remark 2.3.1).

If we can show that equation for $M_T \in L^\infty(\Omega, h)$ then we are done as L^∞ is dense in L^2. Indeed, let $\eta_n \in L^\infty(\Omega, h)$ form a sequence such that $\|\eta_n - M_T\|_{L^2} \to 0$ for $n \to \infty$. Then $\|(Q - \sum_{k=1}^{\infty} Q_k) M_T\|^2 = \|(Q - \sum_{k=1}^{\infty} Q_k)(\lim_n \eta_n)\|^2 = \lim_n \|(Q - \sum_{k=1}^{\infty} Q_k)\eta_n\|^2 = 0$.

So let $M_T \in L^\infty(\Omega, h)$. Then $M_t \in L^\infty(\Omega, h)$ for all $t \leq T$ by a standard argument using the averaging theorem. Thus, again, theorem 2.4.1 supplies the desired σ-additivity. $\square$

Lemma 2.4.4 Stochastic Integral

Consider the assumptions of lemma 2.4.3. Then define a stochastic integral

$$I : L^2(\mathcal{R} \times \Omega, \mathcal{P}, \mu, h) \to L^2(\Omega)$$

via ($\xi \in h$, $\omega \in \Omega$, $s < t$ *in* $\mathcal{R}$, $C \in \mathcal{F}_s$)

$$I(\, 1_{C\times(s,t]}\, \xi\,)(\omega) \;=< \xi, 1_C(\omega)(M_t(\omega) - M_s(\omega)) >_h$$

I *is a contraction. If* $h = \mathbb{C}$ *then* I *is a isometry.*

Proof. This is well-known straightforward stuff, by checking that for $X \in L^2(\mathcal{R}\times\Omega, \mathcal{P}, \mu, h)$ a step function, i.e. $X = \sum_{n=1}^{N} X_n$ where the X_n are pairwise disjoint, simple step functions on a predictable rectangle,

$$\|I(X)\|^2_{L^2(\Omega)} = \sum_{n=1}^{N} \|I(X_n)\|^2_{L^2(\Omega)} \leq \|X\|^2_{L^2(\mathcal{R}\times\Omega,\mathcal{P},\mu,h)}.\Box$$

Similarly as the last lemma one checks

Lemma 2.4.5 *Consider the assumptions of lemma 2.4.3. Then a second kind of integral* ($\omega \in \Omega$, $s < t$ *in* $\mathcal{R}$, $C \in \mathcal{F}_s$)

$$\mathcal{I} : L^2(\mathcal{R}\times\Omega, \mathcal{P}, \mu) \to L^2(\Omega, \mathcal{H}) \;:\; \mathcal{I}(1_{C\times(s,t]}) = 1_C\,(M_t - M_s)$$

can be defined. $\mathcal{I}$ *is a isometry. Especially* $A \mapsto \mathcal{I}(1_A)$, $A \in \mathcal{A}$, *defines a vectorial measure on the algebra* $\mathcal{A} = \{\, p \in \mathcal{P} \mid \mu(p) < \infty\,\}$ *with quadratic variation* μ.

Chapter 3

On annihilated subspaces in $C(K)$

3.1 Introduction

Let K be a compact Hausdorff space and φ a *state* on $C(K)$, i.e. a positive functional with $\varphi(\mathbf{1}) = 1$. Given a linear subspace $\mathcal{H} \subseteq C(K)$ and real constants $c \leq 1 \leq C$ we have worked out a necessary and sufficient criterion on $\mathcal{H}$ to decide, whether there exists a $\mathcal{H}$-annihilating functional ψ with $c\varphi \leq \psi \leq C\varphi$ and $\psi(\mathbf{1}) = 1$, see Theorem 3.2.3. The criterion is remarkably relatively computable. The constraint on ψ can be used to find positive normal $\mathcal{H}$-annihilating functionals (Corollary 3.3.1). For the example $C(K) \cong L^\infty(\mathbb{P})$ ($\mathbb{P}$ a probability measure) Theorem 3.2.3 can be used to decide whether there exists a $\mathcal{H}$-annihilating probability measure $\mathbb{Q}$ with $c \leq d\mathbb{Q}/d\mathbb{P} \leq C$, see (3.16). Due to the nature of criterion (3.16) it would be indeed possible to decide numerically whether a given function $f \in L^\infty(\mathbb{P})$ can be annihilated by a probability measure $\mathbb{Q}$ with the stated bounds on the density functions. In Theorem 3.3.2 we replace the constraint $c \leq d\mathbb{Q}/d\mathbb{P} \leq C$ with $d\mathbb{Q}/d\mathbb{P} \in L^q(\mathbb{P})$, $q > 1$ (but obtain a "non-computable" criterion).

Our result might be interesting in probability theory (searching an equivalent measure under which a stochastic process becomes a martingale, e.g. the pioneering work [6], bounded density functions appear in [16]).

3.2 Annihilation by a similar functional

In what follows K denotes a compact Hausdorff space. In this article all function spaces are real-valued. A *state* on $C(K)$ is a positive linear functional $\varphi : C(K) \to \mathbb{R}$ with $\varphi(\mathbf{1}) = 1$. We denote the maximum/minimum of $f \in C(K)$ by $\max(f)/\min(f)$ and the unit of $C(K)$ by $\mathbf{1}$.

Proposition 3.2.1 *Let K be a compact Hausdorff space, φ a state on $C(K)$, $\mathcal{H} \subseteq C(K)$ a linear subspace and $c \leq 1 \leq C$ reals. Then there exists a linear functional ψ on $C(K)$ with $\psi(\mathbf{1}) = 1$, $c\varphi \leq \psi \leq C\varphi$ and that annihilates $\mathcal{H}$ if and only if for all $x \in \mathcal{H}$*

$$\varphi(x) \leq \inf_{y \in C(K)} (C-1) \max[x + y - \varphi(x+y)\mathbf{1}] + (1-c) \max[y - \varphi(y)\mathbf{1}].$$

Proof. We start with some preparations. Write $C(K) = E \oplus \mathbb{R}\mathbf{1}$ where $E := \ker(\varphi)$. More detailed consider the projection $P : C(K) \to E : Px = x - \varphi(x)\mathbf{1}$ onto E. Then any $x \in C(K)$ is written as $x = Px + \varphi(x)\mathbf{1}$ in our coordinate system. Abbreviate $m : C(K) \to \mathbb{R} : m(x) = -\min(x)$. If we replace in the above large inequality y by $-y$, then it reads

$$\varphi(x) \leq \inf_y (C-1)m(-P(x-y)) + (1-c)m(P(y)). \tag{3.1}$$

The only if part. Now assume that we have given the annihilating functional ψ with the stated properties. Note that clearly $x + m(x)\mathbf{1} \geq 0$ for any $x \in C(K)$. The following two conditions are equivalent.

$$\forall x \in E : c\varphi(x + m(x)\mathbf{1}) \leq \psi(x + m(x)\mathbf{1}) \leq C\varphi(x + m(x)\mathbf{1}) \tag{3.2}$$

$$\forall x \in E : -\psi(x) \leq (1-c)m(x) \text{ and } -\psi(x) \leq (C-1)m(-x) \tag{3.3}$$

Now due to the assumption for any $x \in \mathcal{H}$ we have $0 = \psi(Px) + \varphi(x)$. Hence we get the desired result, i.e. for any $y \in C(K)$

$$\varphi(x) = -\psi(Px) = -\psi(P(x-y)) - \psi(Py)$$

$$\leq (C-1)m(-P(x-y)) + (1-c)m(P(y)).$$

The if part. Note that

$$x \mapsto \alpha(x) := (C-1)m(-x) \qquad \text{and} \qquad x \mapsto \beta(x) := (1-c)m(x)$$

are positive sublinear functionals on E, where the positivity follows from $\max(x) \geq \varphi(x) = 0$. Thus their "convex hull"

$$n(x) := \inf\{\, \alpha(x-y) + \beta(y) \mid y \in E \,\} \qquad (x \in E) \tag{3.4}$$

is also sublinear on E. Now we can define

$$l : P(\mathcal{H}) \to \mathbb{R} : l(Px) = \varphi(x) \tag{3.5}$$

since $Px = 0$ gives $x = \varphi(x)\mathbf{1} \in \mathcal{H}$ and $\varphi(x) \neq 0$ would then contradict the inequality (3.1) by setting $x = y = \mathbf{1}$. Hence by using our assumption we get for $x \in \mathcal{H}$

$$l(P(x)) = \varphi(x) \leq n(Px).$$

Via Hahn-Banach we extend l to a linear map $\hat{l} : E \to \mathbb{R}$ with $\hat{l} \leq n$. Now the functional

$$\psi : C(K) \to \mathbb{R} : \psi(y) = \varphi(y) - \hat{l}(Py) \tag{3.6}$$

obviously annihilates $\mathcal{H}$. For $x \in E$ we get

$$-\psi(x) = \hat{l}(x) \leq n(x) \leq \alpha(x), \beta(x).$$

This is exactly (3.3) and we obtain (3.2). Since $c \leq 1 \leq C$, for real $\lambda \geq 0$ we can add to (3.2) the inequality $c\varphi(\lambda\mathbf{1}) \leq \psi(\lambda\mathbf{1}) \leq C\varphi(\lambda\mathbf{1})$. If we put $\lambda_2 = m(x) + \lambda$ we obtain

$$c\varphi(x + \lambda_2\mathbf{1}) \leq \psi(x + \lambda_2\mathbf{1}) \leq C\varphi(x + \lambda_2\mathbf{1}).$$

Since $x \in E$ and $\lambda_2 \geq m(x)$ were arbitrary, and each positive $y \in C(K)$ has the representation $y = x + \lambda_2\mathbf{1}$ with $x \in E$ and $\lambda_2 \geq m(x)$, we have shown that $c\varphi \leq \psi \leq C\varphi$. The boundedness of ψ clearly follows from the inequality $0 \leq \psi - c\varphi \in C(K)^*$. $\square$

Lemma 3.2.2 *Let K be a compact Hausdorff space, φ a state on $C(K)$, $0 \leq c, C$ reals and $x \in C(K)$. Then*

$$\inf_{y \in C(K)} C \max[x + y - \varphi(x + y)\mathbf{1}] + c \max[y - \varphi(y)\mathbf{1}] \tag{3.7}$$

$$= \min\left\{C\alpha - c\beta \middle| \varphi(x) \leq \alpha \leq \max(x), \beta \leq 0, \varphi((x - \alpha\mathbf{1}) \vee \beta) = 0\right\} - C\varphi(x).$$

The infimum is attained for $y = -[(x - \alpha\mathbf{1}) \vee \beta]$ for some reals $\varphi(x) \leq \alpha \leq \max(x)$ and $\beta \leq 0$ such that $\varphi(y) = 0$.

Proof. For the following computations we will need also discontinuous functions and therefore actually we will consider the larger space $l^\infty(K)$ (all bounded functions on K) instead of $C(K)$. For this reason we extend φ via Hahn-Banach to a state $\tilde{\varphi} \in l^\infty(K)$ ($\tilde{\varphi}$ is a state since $\tilde{\varphi}(\mathbf{1}) = \varphi(\mathbf{1}) = \|\varphi\| = \|\tilde{\varphi}\|$, what already characterizes positivity). Assume that we have shown the lemma for the case that $C(K)$ is replaced by $l^\infty(K)$ and φ is replaced by $\tilde{\varphi}$. Then it is clear that our aimed $C(K)$-version holds too since the infimum in (3.7) is already attained by a continuous function $y \in C(K)$. So we consider $l^\infty(K)$ with a given state φ on it.

Furthermore note that a function in $l^\infty(K)$ clearly need not have a maximum (resp. minimum). In this case we will use the supremum instead of the maximum. Nevertheless we retain our notation and write max (and mean sup). (That is justified in that one could only consider the subspace of functions in $l^\infty(K)$ which attain their maximum and minimum, as one will see in the sequel of this proof.)

We put

$$E := \ker(\varphi) = \{\, y - \varphi(y)\mathbf{1} \mid y \in l^\infty(K) \,\}.$$

We decompose now all vectors in $l^\infty(K)$ into its positive and negative parts where for the positive part we use capital letters and for the negative part we use lower case letters. In order to get the value (3.7) we have to look for $Y - y \in E$ which minimizes the value

$$C \max(X - x + Y - y) + c \max(Y) \tag{3.8}$$

where $X, x, Y, y \geq 0$, $Xx = Yy = 0$. Here we have denoted the expression $x - \varphi(x)\mathbf{1} \in E$ by $X - x$ to get good notations. Note that for arbitrary

$Y-y \in E$ we always have $0 = \varphi(Y-y) \leq \max(Y-y)$, whence all numbers occuring in (3.8) are positive.
Optimization step 1: Let $Y-y \in E$ be given. Let $S(f) = 1_{\{\,\omega\in K \mid f(\omega)\neq 0\,\}}$ denote the support projection of $f \in l^\infty(K)$. Put $z = S(X+Y)y = S(X)y$. We obtain

$$\max(X-x+Y-y) = \max S(X+Y)(X-x+Y-y)$$

$$= \max(X-x+Y-z).$$

Optimization step 2: We put

$$\alpha = 0 \vee \max S(z)(X-z) \qquad \text{and} \qquad w = S(z)(X-\alpha\mathbf{1}) \vee 0. \tag{3.9}$$

Observing that

$$\max S(z)(X-w) = \max[S(z)(X-\alpha\mathbf{1}) - S(z)(X-\alpha\mathbf{1}) \vee 0 + \alpha S(z)] = \alpha \tag{3.10}$$

we proceed

$$\max(X-x+Y-z) = \max S(z)(X-z) \vee \max(\mathbf{1}-S(z))(X-x+Y)$$

$$= \max S(z)(X-w) \vee \max(\mathbf{1}-S(z))(X-x+Y) = \max(X-x+Y-w).$$

Note that $w \leq z \leq y$ since $S(z)X - z \leq \alpha\mathbf{1}$. Put $W = \varphi(w)\varphi(y)^{-1}Y$, so $\varphi(W) = \varphi(w)$, and it is advantageous to replacing $Y-y$ by $W-w \in E$,

$$\max(X-x+Y-y) \geq \max(X-x+W-w) =: \gamma.$$

Optimization step 3: We put

$$v = S(z)(X-\gamma\mathbf{1}) \vee 0.$$

One can calculate

$$\max S(z)(X-v) = \gamma \wedge \max S(z)X. \tag{3.11}$$

Note that $\alpha \leq \gamma$, see (3.10). Thus $v \leq w$ and we put $V = \varphi(v)\varphi(w)^{-1}W$. It is better to replace $W-w$ by $V-v \in E$, indeed

$$\gamma \geq \max S(z)(X-v) \vee \max(\mathbf{1}-S(z))(X-x+V) \tag{3.12}$$

$$= \max(X - x + V - v).$$

Since

$$(\mathbf{1} - S(z))(X - x) \leq (\mathbf{1} - S(z))(X - x + W - w) \leq \gamma\mathbf{1}$$

we obtain

$$(X - x - \gamma\mathbf{1}) \vee 0 = v.$$

If $\gamma \leq \max S(z)X$ we obtain in (3.12) actually an equality, see (3.11). If $\gamma > \max S(z)X$ then $v = V = 0$ and so putting $\gamma_2 := \max(X)$ we get $v = (X - x - \gamma_2\mathbf{1}) \vee 0$ and $\max(X - x + V - v) = \gamma_2$.

Optimization step 4: It became clear that we can restrict us to the following case in the optimization problem (3.8): We have a scalar $0 \leq \alpha \leq \max(X)$ such that $y = (X - x - \alpha\mathbf{1}) \vee 0$ and

$$\max(X - x + Y - y) = \alpha. \tag{3.13}$$

We shall optimize Y. We have

$$0 \geq X - x + Y - y - \alpha\mathbf{1} = (X - x - \alpha\mathbf{1}) + Y - (X - x - \alpha\mathbf{1}) \vee 0.$$

Therefore we obtain the following necessary conditions on Y:

$$0 \leq Y \leq -[(X - x - \alpha\mathbf{1}) \wedge 0]. \tag{3.14}$$

On the other hand if some Y satisfies (3.14), then $Yy = 0$ and equality (3.13) holds further. Thus we can feel free to choose any Y satisfying (3.14) without touching the left summand in (3.8) and we just have to optimize $\max(Y)$.

Therefore it is clear that the best choice is to simply cut the function $-[(X - x - \alpha\mathbf{1}) \wedge 0]$, i.e. to choose

$$Y = -[(X - x - \alpha\mathbf{1}) \wedge 0 \vee \beta]$$

for some $\beta \leq 0$ and we obtain $\max(Y) = -\beta$. To be precise such a β exists since for arbitrary $z \in A$ the function $\mathbb{R} \to \mathbb{R} : \beta \mapsto \varphi(z \vee \beta)$ is continuous and hence the necessary condition

$$0 = \varphi(y - Y) = \varphi((X - x - \alpha\mathbf{1}) \vee \beta)$$

has a solution for some $\beta \leq 0$. So $Y - y$ is now of the desired form and (3.8) becomes $C\alpha - c\beta$. *Final step:* Therefore (3.7) is

$$\inf \left\{ C\alpha - c\beta \middle| 0 \leq \alpha \leq \max(x), \beta \leq 0, \varphi((X - x - \alpha\mathbf{1}) \vee \beta) = 0 \right\}.$$

It is clear that the infimum is attained here, as (α, β) can be chosen from a compact set. More detailed $(\alpha, \beta) \in [0, \max(x)] \times [\min(x), 0] \cap \ker(f)$ where f is the continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : f(\alpha, \beta) = \varphi((x - \alpha\mathbf{1}) \vee \beta)$. We finish the proof by abandoning the early "substitution" $x - \varphi(x)\mathbf{1} \equiv X - x$. $\square$

If we combine Proposition 3.2.1 with Lemma 3.2.2 then we obtain

Theorem 3.2.3 *Let K be a compact Hausdorff space, φ a state on $C(K)$, $\mathcal{H} \subseteq C(K)$ a linear subspace and $c \leq 1 \leq C$ reals. Then the following conditions are equivalent:*

1. $\forall x \in \mathcal{H} : \varphi(x) \leq$
$$C^{-1} \min \left\{ (C-1)\alpha + (c-1)\beta \middle| \varphi(x) \leq \alpha \leq \max(x), \beta \leq 0, \varphi((x-\alpha) \vee \beta) = 0 \right\}.$$
2. *There exists a functional $\psi \in C(K)^*$ with $\psi(\mathcal{H}) = 0$, $\psi(\mathbf{1}) = 1$ and $c\varphi \leq \psi \leq C\varphi$.*

Note that the criterion (1) in the above theorem is positively homogenous, i.e. if it is fulfilled for x then it is fulfilled for all λx, $\lambda \geq 0$.

3.3 Annihilation by faithful normal states

A functional $\varphi \in C(K)^*$ is said to be *normal* if it is continuous with respect to the pointwise convergence on the unit ball of $C(K)$. I.e. φ is normal if for all nets $(f_i) \subseteq C(K)$ with the properties $\forall i : \|f_i\|_{C(K)} \leq 1$ and $\forall \omega \in K : f_i(\omega) \to 0$ we have $\varphi(f_i) \to 0$.

A state φ is *faithful* if $\varphi(x) > 0$ for all $x > 0$.

A compact Hausdorff space K is said to be *stonean* if the closure of every open subset is open. A *projection* $f \in C(K)$ is an element with $ff = f$. So any projection in $C(K)$ is of the form 1_Y for some open and closed subset $Y \subseteq X$.

Corollary 3.3.1 *Let K be a compact Hausdorff space and $\varphi \in C(K)^*$ be a faithful normal state on $C(K)$. Let $\mathcal{H} \subseteq C(K)$ be a linear subspace and $0 < c \leq 1 \leq C$ reals.*

1. *There exists a faithful normal state $\psi \in C(K)^*$ such that $\psi(\mathcal{H}) = 0$ and $c\varphi \leq \psi \leq C\varphi$ if and only if criterion Theorem 3.2.3.(1) is fulfilled.*

2. *Let K be stonean. There exists a faithful normal state $\psi \in C(K)^*$ such that $\psi(\mathcal{H}) = 0$ and $\psi \leq C\varphi$ if and only if for each nonzero projection $f \in C(K)$ there exists a nonzero projection $e \leq f$ and a real $\gamma > 0$ such that*
$$\forall x \in \mathcal{H} \oplus [e - \gamma \mathbf{1}] : \varphi(x) \leq$$
$$C^{-1} \min \Big\{ (C-1)\alpha - \beta \,\Big|\, \varphi(x) \leq \alpha \leq \max(x), \beta \leq 0, \varphi((x-\alpha) \vee \beta) = 0 \Big\}.$$

Proof. (1) This is Theorem 3.2.3. The faithfulness and normality of ψ follows immediately from $c\varphi \leq \psi \leq C\varphi$.

(2) *The only if part.* Assume we have that functional ψ. Fix a nonzero projection $f \in C(K)$. Then $\psi(f - \psi(f)\mathbf{1}) = 0$, where $\psi(f) > 0$ since ψ is faithful. Hence we can apply Theorem 3.2.3 for $c = 0$ and for the subspace $\mathcal{H} \oplus [e - \psi(e)\mathbf{1}]$.

The if part. At first we note that due to Theorem 3.2.3 ($c = 0$) and the assumptions, for any nonzero projection f we find a nonzero projection $e \leq f$, a $\gamma > 0$ and a (normal) state $0 \leq \psi \leq C\varphi$ such that $\psi(\mathcal{H} \oplus [e - \gamma\mathbf{1}]) = 0$. In particular $\psi(e) = \gamma > 0$ and $\psi(\mathcal{H}) = 0$.

Consider the normal states $\psi \in C(X)^*$ such that $\psi(\mathcal{H}) = 0$ and such that $\psi \leq C\varphi$. We call them *solutions* here. Denote by $s(\psi) \in C(K)$ the support projection of a normal functional ψ ([17, III.1.13 or III.3.7]). Following the idea of Kreps [9] or Yan [18], we can show that there is a most faithful solution in the sense that it has maximal support projection within all solutions. Indeed consider

$$\beta = \sup\{\, \varphi(s(\psi)) \mid \psi \text{ is a solution} \,\}. \tag{3.15}$$

Let ψ_n a sequence of solutions such that $\beta = \lim_n \varphi(s(\psi_n))$. Set $\psi = \sum_{n=1}^{\infty} 2^{-n}\psi_n$, where the convergence is in the norm of $C(X)^*$. Clearly ψ

is then also a solution. Thus

$$\beta \geq \varphi(s(\psi)) \geq \lim_{n\to\infty} \varphi(s(\psi_n)) = \beta.$$

Hence the supremum in (3.15) is attained at ψ. Now assume $s(\psi) \neq \mathbf{1}$. Then due to what we have said above, for $\mathbf{1} - s(\psi)$ we find a nonzero projection $e \leq \mathbf{1} - s(\psi)$ and a solution π such that $\pi(e) > 0$. This is a contradiction since $(\pi + \psi)/2$ would then be a solution such that $\varphi(s(\pi + \psi)/2) > \beta$. We conclude that ψ is a faithful solution. $\square$

We review now our last results under the light of probability measures. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we have an isometric isomorphism $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \cong C(K)$ via Gelfand's transformation and where K is stonean ([17, I.4.4] or [7, 5.2.1]). (Alternatively, all proofs would also go through for $L^\infty(\mathbb{P})$ if one replaces $\max(f)$ by ess $\sup(f)$. So actually the Gelfand representation is not really necessary.) Consider the normal functional $\varphi : L^\infty(\mathbb{P}) \to \mathbb{R} : \varphi(f) = \int f d\mathbb{P}$. The normal functionals on $L^\infty(\mathbb{P})$ are just the space $L^1(\mathbb{P}) \subseteq (L^\infty(\mathbb{P}))^*$. (Alternatively, define the space of normal functionals simply as the $L^1(\mathbb{P})$ or, equivalently and like above, as the functionals which are continuous on the unit ball of $L^\infty(\mathbb{P})$ w.r.t. the pointwise convergence.) Then, in example, Corollary 3.3.1.(1) could be restated as follows: Let $0 < c \leq 1 \leq C$. Then there exists a equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ that annihilates $\mathcal{H}$ and such that $c \leq d\mathbb{Q}/d\mathbb{P} \leq C$ if and only if

$$\forall x \in \mathcal{H} : \int x d\mathbb{P} \leq C^{-1} \min \Big\{ (C-1)\alpha + (c-1)\beta \Big| \tag{3.16}$$

$$\int x d\mathbb{P} \leq \alpha \leq \text{ess sup}(x), \beta \leq 0, \int (x - \alpha) \vee \beta d\mathbb{P} = 0 \Big\}.$$

(The equivalence $\mathbb{Q} \sim \mathbb{P}$ is meant in the sense of absolute continuity.) We have also the following similar results with other properties on the density function $d\mathbb{Q}/d\mathbb{P}$.

Theorem 3.3.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ a linear subspace and $0 < c \leq 1 \leq C$, $1 < q \leq \infty$ reals. Let p be defined by $1/p + 1/q = 1$.*

1. *There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\int \mathcal{H} d\mathbb{Q} = 0$, $d\mathbb{Q}/d\mathbb{P} \in L^q(\mathbb{P})$ and $c \leq d\mathbb{Q}/d\mathbb{P}$ if and only if there exists a real $D \geq 0$ such that $\forall x \in \mathcal{H} \forall y \in L^\infty(\mathbb{P})$*

$$\int x d\mathbb{P} \leq D \Big\| x + y - \int (x+y) d\mathbb{P} \mathbf{1} \Big\|_{L^p(\mathbb{P})} + (1-c) \sup \Big(y - \int y d\mathbb{P} \mathbf{1} \Big).$$

2. *There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\int \mathcal{H} d\mathbb{Q} = 0$ and $d\mathbb{Q}/d\mathbb{P} \in L^q(\mathbb{P})$ if and only if for each nonzero projection $f \in L^\infty(\mathbb{P})$ there exist a nonzero projection $e \leq f$ and reals $\gamma > 0$, $D \geq 0$ such that $\forall x \in \mathcal{H} \oplus [e - \gamma \mathbf{1}] \forall y \in L^\infty(\mathbb{P})$*

$$\int x d\mathbb{P} \leq D \Big\| x + y - \int (x+y) d\mathbb{P} \mathbf{1} \Big\|_{L^p(\mathbb{P})} + \sup \Big(y - \int y d\mathbb{P} \mathbf{1} \Big).$$

Proof. (1) The proof is very similar to that of 3.2.1. For $x \in L^\infty(\mathbb{P})$ we put

$$\varphi(x) = \int x d\mathbb{P} \qquad \psi(x) = \int x d\mathbb{Q}$$

$$E = \ker(\varphi) \qquad m(x) = \text{ess sup}(-x) \qquad P(x) = x - \varphi(x)\mathbf{1}.$$

Note that $E = P(L^\infty(\mathbb{P}))$ and that the occuring large inequality, after replacing y by $-y$, reads

$$\varphi(x) \leq D \|P(x-y)\|_{L^p(\mathbb{P})} + (1-c) m(Py). \tag{3.17}$$

The only if part. Assume that we have that measure ψ. Firstly, for arbitrary $x \in L^\infty(\mathbb{P})$,

$$-\psi(x) \leq |\psi(x)| = \Big| \int x \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} \Big| \leq D \|x\|_{L^p(\mathbb{P})}, \tag{3.18}$$

where $D := \|d\mathbb{Q}/d\mathbb{P}\|_{L^q(\mathbb{P})}$. Note that our assumption $c \leq d\mathbb{Q}/d\mathbb{P}$ yields $c\varphi \leq \psi$, and that $x + m(x)\mathbf{1} \geq 0$ for any $x \in E$. Thus, for any $x \in E$, $c\varphi(x + m(x)\mathbf{1}) \leq \psi(x + m(x)\mathbf{1})$. For $x \in E$ this yields

$$-\psi(x) \leq (1-c) m(x). \tag{3.19}$$

Now for any $x \in \mathcal{H}$ we have $0 = \psi(x) = \psi(Px) + \varphi(x)$. Thus we get the desired result (3.17) by combining (3.18) and (3.19), for $x \in \mathcal{H}$ and $y \in L^\infty(\mathbb{P})$

$$\begin{aligned} \varphi(x) &= -\psi(Px) = -\psi(P(x-y)) - \psi(Py) \\ &\leq D\|P(x-y)\|_{L^p(\mathbb{P})} + (1-c)m(Py). \end{aligned}$$

The if part. We omit some details and request the reader to proceed as in the proof of Prop. 3.2.1. Define sublinear functionals α, β on E,

$$\alpha(x) = D\|x\|_{L^p(\mathbb{P})} \qquad \text{and} \qquad \beta(x) = (1-c)m(x),$$

and consider their convex hull n (3.4). We define l as in (3.5). Taking our assumption (3.17) into account we get for $x \in L^\infty(\mathbb{P})$

$$l(Px) = \varphi(x) \leq \inf_{y \in L^\infty(\mathbb{P})} \alpha(P(x-y)) + \beta(Py) = n(Px). \tag{3.20}$$

We extend l to $\hat{l}$ via Hahn-Banach and proceeding analogously as in the proof of 3.2.1 we obtain the desired state $\psi : L^\infty(\mathbb{P}) \to \mathbb{R}$ that annihilates $\mathcal{H}$ and satisfies $c\varphi \leq \psi$.

It remains to show that ψ defines a measure $\mathbb{Q}$ with density in $L^q(\mathbb{P})$. Consider the map $\phi = \varphi | L^p(\mathbb{P})$. We have $L^p(\mathbb{P}) \cong ker(\phi) \oplus \mathbb{R}$ (Banach space isomorphically) since ϕ is a bounded projection onto the second coordinate. Therefore we find a constant $M > 0$ such that for all $x + a\mathbf{1} \in L^\infty(\mathbb{P})$ ($x \in E$, $a \in \mathbb{R}$) we have

$$|\psi(x + a\mathbf{1})| \leq |\hat{l}(x)| + a \leq D\|x\|_{L^p(\mathbb{P})} + a \leq M\|x + a\mathbf{1}\|_{L^p(\mathbb{P})}.$$

(Recall that ψ is defined via (3.6) and that $\hat{l}$ also satisfies inequality (3.20).) Thus ψ can be extended continuously on $L^p(\mathbb{P})$ and we obtain $\hat{\psi} \in L^p(\mathbb{P})^* \subseteq L^q(\mathbb{P})$. Therefore for all $x \in L^\infty(\mathbb{P})$, $\psi(x) = \hat{\psi}(x) = \int xV d\mathbb{P}$ for some $V \in L^q(\mathbb{P})$.

(2) The above proof of (1) goes also through for $c = 0$. But the "if part" has the following weakness: One finds ψ such that $0 = c\varphi \leq \psi$ and hence ψ need not necessarily be equivalent to φ. But due to the assumption for any nonzero projection f we find a nonzero projection $e \leq f$ and a $\mathcal{H}$-annihilating state ψ with $\psi(e) > 0$ and such that the density function $d\mathbb{Q}/d\mathbb{P} \cong \psi$ is in

$L^q(\mathbb{P})$. Arguing now as in the proof of Corollary 3.3.1.(2) one can construct a faithful solution ψ. (The *solutions* let be the $\mathcal{H}$-annihilating states with density function in $L^q(\mathbb{P})$.) $\square$

In the next corollary we use the easy fact that the discussed question whether there exists a $\mathcal{H}$-annihilating and $\mathbb{P}$-equivalent measure $\mathbb{Q}$ under constraint, is "dense" in a certain sense within the cases where we have no restriction on the density function.

Corollary 3.3.3 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{H} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ a linear subspace and $\varepsilon > 0$. Then the following conditions are equivalent.*

1. *There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\int \mathcal{H} d\mathbb{Q} = 0$.*
2. *There exist a probability measure $\mathbb{R} \sim \mathbb{P}$ and reals $0 < c \leq 1 \leq C$ such that $\|d\mathbb{R}/d\mathbb{P} - \mathbf{1}\|_{L^1(\mathbb{P})} \leq \varepsilon$ and criterion (3.16) is fulfilled if one replaces $\mathbb{P}$ with $\mathbb{R}$.*
3. *For all measurable sets $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ there exist a $\mathbb{P}$-absolute continuous probability measure $\mathbb{R}$ with $\mathbb{R}(A) > 0$ and reals $0 < c \leq 1 \leq C$ such that criterion (3.16) is fulfilled if one replaces $\mathbb{P}$ with $\mathbb{R}$.*

Proof. (1) $\Rightarrow$ (2): Let φ and $\psi \in (L^\infty(\mathbb{P}))^*$ be associated to $\mathbb{P}$ and $\mathbb{Q}$. Define $a\varphi(x) := \varphi(xa)$ for $a, x \in L^\infty(\mathbb{P})$. Fix real constants $0 < m \leq 1 \leq M$ and set

$$A = \{\, \omega \in \Omega \mid m \leq d\mathbb{Q}/d\mathbb{P}(\omega) \leq M \,\}.$$

Set $\alpha = \varphi(1_A) + \psi(1_{A^C})$ and $\tilde{\varphi} = (1_A\varphi + 1_{A^C}\psi)\alpha^{-1}$. Now for any $x \geq 0$ we have

$$\begin{aligned}\psi(x) = \psi(x1_A + x1_{A^C}) &= \varphi(x1_A d\mathbb{Q}/d\mathbb{P}) + \psi(x1_{A^C}) \\ &\leq M\varphi(x1_A) + M\psi(x1_{A^C}) = M\tilde{\varphi}(x)\alpha.\end{aligned}$$

Analogously we have $\psi(x) \geq m\tilde{\varphi}\alpha$. Thus if $\mathbb{R}$ denotes the probability measure associated to $\tilde{\varphi}$ then $\alpha m \leq d\mathbb{Q}/d\mathbb{R} \leq \alpha M$ and it is clear that we have (3.16) if we replace $\mathbb{P}$ with $\mathbb{R}$. It is easy to see that $\|d\mathbb{R}/d\mathbb{P} - \mathbf{1}\|_{L^1(\mathbb{P})} \to 0$ if $m \to 0$ and $M \to \infty$. (2) $\Rightarrow$ (3): Obviously. (3) $\Rightarrow$ (1): For a given nonzero projection $1_A \in L^\infty(\mathbb{P})$ we find a $\mathcal{H}$-annihilating measure $\mathbb{Q} \sim \mathbb{R}$

such that $\int 1_A d\mathbb{Q} > 0$. Arguing now as in the proof of Corollary 3.3.1.(2) one can construct a $\mathcal{H}$-annihilating measure $\mathbb{Q} \sim \mathbb{P}$. (The *solutions* let be the $\mathcal{H}$-annihilating normal states.) $\Box$

Acknowledgement

I would like to thank my superviser Prof. Gerhard Larcher for his financial support and for the liberty in doing my research. Due to his tolerance this work could be done in a relaxed atmosphere. I would also like to thank my second superviser Prof. Reinhard Wolf for this task. Furthermore I thank and greet my nice working colleges Gunther, Fritz, Lucia, Judith and Klaus. I thank Gunther for correction reading parts of Chapter 2, for valuable discussions and for arranging contacts to Prof. Emery. I am greatful to Prof. Emery for his support. I am indebted to Prof. Cooper for correction reading Chapter 3. I thank Fritz for drawing Figure 1.1 and for his support with LaTeX.

Curriculum Vitae

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